Universidad Autónoma de Madrid
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Nonlinear and Nonlocal Diffusion Equations

Memoria para optar al título de Doctor en Matemáticas

por

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Summary

In this thesis we consider three different models of nonlinear diffusion evolution equations of parabolic type. Such equations model various diffusion phenomena. The prototype is the classical Porous Medium Equation

$$u_t = \Delta u^m, \tag{PME}$$

which models the flow of gasses through porous media. Starting from the classical PME, different directions can be considered, due to their recent interest for the mathematical community and their applicability in modelling physical events, as we will describe in Section 0.3.

My contribution to the material presented in this thesis is contained in the papers [94–98] in collaboration with my advisor, Professor Juan Luis Vázquez, and [94–96] coauthored also with my colleague Félix del Teso.

The main topics of the thesis are the following:

(1) *The asymptotic behaviour of the doubly nonlinear diffusion equation* $u_t = \Delta_p(u^m)$ *on bounded domains* with $m > 0$, $p > 1$. In [98] we prove sharp rates of convergence to the asymptotic profile in the degenerate case $m(p-1) > 1$ and convergence in relative error in the quasilinear case $m(p-1) = 1$. The results are new even in the particular case $m = 1$, $p > 1$.

(2) *The Fisher-KPP equation with nonlinear fractional diffusion* (KPP) $u_t + (-\Delta)^s(u^m) = u(1-u)$ with $s \in (0,1)$ and $m > (N-2s)/N$. The study of this problem is connected to the Fractional Porous Medium Equation (FPME) $u_t + (-\Delta)^s(u^m) = 0$. The asymptotic behavior of the fractional model (KPP) with $s \in (0,1)$ differs from the one of the local case $s = 1$ which has a linear propagation of level sets and admits linear traveling wave solutions. In [97] we prove that, for initial data with suitable decay, the fractional model does not admit linear traveling-wave solutions since the level sets propagate exponentially fast in time. In the limit $m \to 1$ we recover the linear fractional case $m = 1$, $s \in (0,1)$, studied by Cabré and Roquejoffre [37].

(3) *Porous Medium with fractional pressure (PMFP)*: $u_t = \nabla(u^{m-1}\nabla P)$, with $P = \mathcal{K}(u)$, $\mathcal{K}$ being a Riesz potential. The problem has been recently studied by Caffarelli, Vázquez [39] and Biler, Karch, Monneau [22] in the particular case $m = 2$, in which they show the relevance of this model for applications. In [94, 95], we investigate the effect of the nonlinearity on the finite speed of propagation of the solution. More precisely, we prove two different behaviors depending on the exponent m: finite speed of propagation when $m \geq 2$, infinite speed of propagation when $m \in (1,2)$.

In a following paper [96] we prove a correspondence between self-similar solutions for (FPME) and (PMFP). This result represents a great progress in understanding and connecting the two theories of nonlocal Porous Medium type presented nowadays in the literature.

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La tesis se realizó en el programa de doctorado del Departamento de Matemáticas de la Universidad Autónoma de Madrid. Durante los cuatro años del doctorado, me he beneficiado de las siguientes fuentes de financiación:
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To my friends.
Also, I would like to express my gratitude to my family and to Cornel.

Contents

Introducción y presentación de resultados

La *Ecuación del Calor* es la mas sencilla ecuación que describe procesos de difusión:

$$u_t = \Delta u \tag{HE}$$

donde $u(x,t)$ denota la concentración/densidad de una sustancia. Esta descripción fue primero dada por Fourier en 1822 en su célebre libro *Théorie Analytique de la Chaleur* [62]. La Ecuación del Calor está relacionada con el estudio del movimiento Browniano de partículas y representa el modelo básico de difusión lineal, aparecido en el pionero artículo de Einstein [56].

Existe una amplia clase de sistemas físicos que no se pueden describir por Ecuaciones en Derivadas Parciales (EDP) lineales. Las ecuaciones en derivadas parciales no lineales son la herramienta moderna utilizada para describir muchos fenómenos físicos en dinámica de fluidos, dinámica de poblaciones, elasticidad, relatividad y termodinámica. Una clase importante de EDPs no lineales son las ecuaciones parabólicas de segundo orden, usadas para describir procesos de difusión de la clase anterior bajo condiciones distintas, haciendo de alguna manera el modelo más realista.

El estudio de las ecuaciones en derivadas parciales no lineales es bastante complejo. En esta tesis presentaremos tres diferentes modelos de difusión no lineal que requieren el uso de técnicas diferentes. En general, $u(x,t)$ denota la densidad de un líquido o población en un tiempo $t \geq 0$ y coordenada espacial $x \in \mathbb{R}^N$. La función u es solución de una EDP parabólica de las mencionadas anteriormente. Estamos interesados en los siguientes problemas: existencia y unicidad de soluciones u y su comportamiento asintótico para tiempos grandes. En particular, el estudio de soluciones autosemejantes y soluciones tipo onda viajera nos da una mejor comprensión de propiedades tales como convergencia (con tasas) a un perfil estacionario, estimaciones de positividad o aparición de fronteras libres.

0.1 Aplicaciones físicas

0.1.1 Modelo físico

El modelo básico que vamos a estudiar fue derivado de manera independiente por Leibenzon y Muskat alrededor de 1930 en el estudio del flujo de un gas isentrópico a través de un medio poroso. También fue considerado anteriormente en el estudio de la infiltración de aguas subterráneas por Boussisnesq en 1903.

Consideramos un medio continuo (líquido o población) representado por una densidad de distribución $u(x,t) \geq 0$ que evoluciona con el tiempo, según un campo de velocidades $\mathbf{v}(x,t)$ correspondiente a la ecuación de continuidad

$$u_t + \nabla \cdot (u \cdot \mathbf{v}) = 0.$$

(i) Los fluidos en medios porosos se comportan conforme a la Ley de Darcy: la velocidad $\mathbf{v}$ proviene de un potencial

$$\mathbf{v} = -\nabla P,$$

donde P denota la presión.

(ii) Relación entre P y u: para gases en medios porosos, Leibenzon [83] y Muskat [87] (1930) obtuvieron una relación en la forma de la siguiente ley de estado:

$$P = f(u),$$

donde f es una función escalar no-decreciente. Esta función $f(u)$ es lineal cuando el flujo es isotérmico y es una potencia superior de u cuando el flujo es adiabático, i.e. $f(u) = cu^{m-1}$ con $c > 0$ y $m > 1$.

La dependencia lineal $f(u) = cu$ fue también obtenida por Boussinesq [33] en 1903 modelando la infiltración del agua en una capa de tierra casi horizontal, en cuyo caso $u_t = c/2\, \Delta u^2$.

Por lo tanto, el modelo puede ser escrito en forma $u_t = (c/m)\, \Delta u^m$, o, después de un reescalamiento en la variable de tiempo,

$$u_t = \Delta u^m. \tag{PME}$$

Esta ecuación es conocida como la *Ecuación de medios porosos*. Es el modelo típico de difusión no lineal , el cual es el punto de partida para esta tesis. Véase libro de Vázquez [106] para más detalles.

0.1.2 Difusión no lineal

Dos de los modelos de difusión no lineales mas populares son los siguientes:
(i) La Ecuación de medios porosos con $m > 1$

$$u_t = \Delta u^m. \tag{PME}$$

(ii) La ecuación de evolución p-Laplaciana, con $p > 1$,

$$u_t = \Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u). \tag{PLE}$$

La (PME) y la (PLE) son ambas ecuaciones no lineales de tipo parabólico, perteneciendo a la clase

$$u_t = \nabla \cdot (A(u, \nabla u)\nabla u).$$

La (PME) tiene *difusividad dependiente de la densidad*: el coeficiente de difusión $A(u) = mu^{m-1}$ hace que la ecuación sea degenerada para $m > 1$ y singular para $m < 1$. La (PLE) pertenece a la clase de ecuaciones con *difusión dependiente del gradiente*: $A(\nabla u) = |\nabla u|^{p-2}$, que degenera para $p > 2$ y es singular para $p < 2$. En el caso particular $m = 1$, respectivamente $p = 2$, recuperamos la *Ecuación del Calor* (HE) clásica.

Cuando $m < 1$ la (PME) es llamada *La ecuación de difusión rápida*, y tiene aplicaciones en física del plasma. John King, [76], también obtuvo la ecuación de difusión rápida para describir la difusión de impurezas en el silicio.

La terminología *difusión lenta/rápida* se refiere a la propiedad de velocidad finita/infinita de propagación, la cual depende del rango de los parámetros $m > 1$ y $m < 1$ para la (PME), respectivamente $p > 2$ y $1 < p < 2$ para la (PLE).

La ecuación p-laplaciana (PLE) se utiliza para describir la filtración a través de un medio poroso de un fluido no-newtoniano, ver el libro de Ladyzhenskaya [81] sobre los flujos viscosos incompresibles.

Las ecuaciones de difusión tienen una amplia aplicabilidad en tratamiento de imágenes. Por ejemplo, podemos mencionar: la eliminación del ruido de la imagen basado en ecuaciones de difusión anisotrópica (el modelo Perona-Malik [88]), aplicación del PLE rápida ($p < 2$) por Barenblatt y Vázquez para mejora del contorno del imagen [14]. También mencionamos: Barbu [12], Weickert [113], Catté et al. [44].

Mediante la combinación de ambos tipos de nolinealidades, obtenemos la *Ecuación de Difusión con Doble Nolinealidad*

$$u_t = \Delta_p u^m \tag{DNLE}$$

con $m > 0$, $p > 1$. Este modelo general hereda propiedades de las (PME) y PLE.

Estos modelos han sido profundamente estudiados; podemos mencionar varios trabajos de Vázquez: la monografía [106] para la teoría de la (PME), el libro [105] sobre estimaciones y smoothing, el survey [104] para el comportamiento asintótico. La teoría de ecuaciones parabólicas comenzó con el trabajo de Ladyzhenskaya, Solonnikov y Ural'seva [80]. Más recientemente, podemos mencionar el libro de DiBenedetto [55].

0.1.3 Difusión no local

Los procesos de difusión con efectos de largo alcance son modelados en muchas situaciones utilizando operadores no locales. Es de especial interés en esta tesis el operador Laplaciano Fraccionario (Stein [99], Landkof [82]), que generalmente se define mediante la transformada de Fourier

$$\mathcal{F}((-\Delta)^s f(x)) = |\xi|^{2s}\mathcal{F}(f)(\xi),$$

para funciones f en la clase de Schwartz. Cuando $0 < s < 1$, el operador Laplaciano Fraccionario también puede ser definido por la siguiente fórmula integral utilizando núcleos hiper-singulares

$$(-\Delta)^s f(x) = C_{N,s}\,\text{P.V.} \int_{\mathbb{R}^N} \frac{f(x)-f(y)}{|x-y|^{N+2s}}dy$$

donde P.V. representa el valor principal y $C_{N,s} = \pi^{-(2s+N/2)}\Gamma(N/2+s)/\Gamma(-s)$ es una constante de normalización. Cuando $s \to 0$ recuperamos el operador identidad, mientras que para $s \to 1$ recuperamos el Laplaciano estándar. El operador inverso es dado por el potencial de Riesz

$$(-\Delta)^{-s} f(x) = C_{N,-s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}}dy.$$

La *Ecuación del Calor Fraccionaria*

$$u_t + (-\Delta)^s u = 0$$

modela casos de difusión con efectos de largo alcance presente en una serie de fenómenos en finanzas, física, biología, [...].

En la probabilidad, el operador Laplaciano Fraccionario es el generador infinitesimal de procesos estables de Levy [7, 18, 100]. El uso de operadores fraccionarios en la modelización de procesos de difusión ha sido conocido recientemente por el trabajo de Caffarelli, Vázquez, Athanasopoulos, Salsa, Silvestre, Kassmann, Cabré, Valdinoci y muchos otros, la mayoría de los temas sobre problemas estacionarios.

La difusión anómala es a menudo descrita por modelos no lineales con operadores fraccionarios. Presentamos ahora dos modelos de flujos en medios porosos que implican efectos de difusión no local .

(i) (PME) No Local. El primer modelo presentado es la Ecuación de Medios Porosos Fraccionaria $m > 1$, $s \in (0,1)$

$$u_t + (-\Delta)^s u^m = 0 \tag{FPME}$$

donde $(-\Delta)^s$ es el operador Laplaciano Fraccionario descrito anteriormente. Este modelo puede ser interpretado como el correspondiente no local del clásico PME ($s = 1,\ m > 1$). Las principales diferencias aparecen en el nivel de propagación: por un lado la (PME) tiene velocidad de propagación finita, que generalmente se expresa demostrando que datos iniciales de soporte compacto producen soluciones con soporte compacto (por lo tanto, aparecen fronteras libres). Por otro lado, su correspondiente no local, la (FPME) tiene velocidad de propagación infinita, es decir, incluso si u_0 tiene soporte compacto, la solución es estrictamente positiva para todos los tiempos $u(x,t) > 0$, $t > 0$. Esta última propiedad es una consecuencia del carácter no local del operador para $s \in (0,1)$. La (FPME) ha sido estudiada por Vázquez y colaboradores [28, 46–48] y otros.

Las ecuaciones de reacción-difusión aparecen en la modelización de difusión combinada con fenómenos de reacción. Cuando el término de la reacción es de la forma $u(1-u)$, el modelo es conocido como *Fisher-KPP* que fue primero usado para describir la propagación de una población biológica por Fisher [60] y Kolmogorov, Petrovskii y Piskunov [77]. En la siguiente sección 0.2.2 se describe el modelo de difusión fraccionaria no lineal combinada con reacción de tipo Fisher-KPP $u_t + (-\Delta)^s u^m = u(1-u)$.

(ii) (PME) con presión fraccionaria. Consideremos el caso en que la presión P esta relacionada con la densidad u por medio de un operador no local de tipo fraccionario

$$P = \mathcal{K}(u),$$

donde $\mathcal{K} = (-\Delta)^{-s}$ es el inverso del operador Laplaciano Fraccionario, $0 < s < 1$. El modelo de difusión con efectos no locales es el siguiente

$$u_t = \nabla \cdot (u \nabla P), \quad P = (-\Delta)^{-s}(u). \tag{CV}$$

La ecuación ha sido introducida por Caffarelli y Vázquez [39], sus propiedades han sido estudiadas en una serie de artículos [38, 40]. Por otro lado, un modelo similar fue propuesto por Head [67] para describir la dinámica de la dislocación en cristales, vistos como medio continuo [67]. En el limite $s \to 1$ (Serfaty y Vázquez en [91]) resulta una ecuación tipo "mean field" presente en la superconductividad y superfluidez.

Partiendo de la ecuación (CV), se pueden considerar modelos más generales que mantienen las principales características de la (PME). Aquí discutimos el siguiente problema

$$u_t = \nabla \cdot (u^{m-1} \nabla P), \quad P = (-\Delta)^{-s}(u). \tag{PMFP}$$

Llamamos a este problema *Ecuación de Medios Porosos con presión fraccionaria*, brevemente (PMFP). La nolinealidad tiene efectos sobre la propiedad de propagación de la solución, como describiremos en la sección 0.2.3.

0.2 Resultados principales

Presentamos los resultados obtenidos por la autora en la elaboración de la presente tesis doctoral.

0.2.1 La Ecuación de Difusión con Doble Nolinealidad

Consideramos el problema

$$u_t = \Delta_p u^m \tag{DNLE}$$

donde $m > 0$, $p > 1$ y $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ es el operador $p-$Laplaciano.

Cuando $p = 2$ recuperamos la ecuación de medios porosos $u_t = \Delta u^m$ con $u_t = \Delta u^m$; al mismo tiempo, cuando $m = 1$, recuperamos la ecuación p-laplaciana degenerada (PLE) $u_t = \Delta_p u$ con $p > 2$, ambas ecuaciones bien conocidas en la literatura.

Como referencias para la teoría anterior para la (DNLE) mencionamos: el comportamiento asintótico del problema de Cauchy por Carrillo, Agueh, Blanchet [2], existencia y unicidad del problema de Dirichlet en un dominio acotado por Manfredi, Vespri, Savaré [86, 90], estimaciones por Bonforte, Grillo [30], regularidad por Vespri, Porzio [89], Kuusi, Urbano [79]. Para la regularidad de la ecuación p-laplaciana mencionamos el trabajo de Kuusi y Mingione [78]. Vease también Bonforte, Di Castro [29] sobre estimaciones cuantitativas locales para ecuaciones elípticas no lineales que involucran operadores de tipo p-Laplaciano. Sobre la regularidad en el caso singular mencionamos un artículo reciente de Fornaro, Sosio y Vespri [61].

Investigamos el problema de Dirichlet homogéneo en un dominio acotado $\Omega \subset \mathbb{R}^N$ con datos iniciales $u(x,0) = u_0(x)$ en el rango de parámetros $m(p-1) \geq 1$. Para dato nonegativo e integrable u_0, estudiamos el comportamiento asintótico de la solución positiva $u(x,t)$ del problema (DNLE). Muchos de nuestros resultados son nuevos también en el caso de la ecuación p-Laplaciana $m = 1$, $p > 2$.

(I) El caso degenerado $m(p-1) > 1$. Vamos a demostrar tasas de convergencia de la solución rescalada $v(x,\tau) = t^{\frac{1}{m(p-1)-1}} u(x,t)$, $t = e^\tau$ a un único perfil asintótico $f(x)$, cuando $\tau \to +\infty$. El perfil estacionario f se caracteriza como la solución positiva del problema estacionario correspondiente

$$\Delta_p f^m + \frac{1}{m(p-1)-1} f = 0 \text{ en } \Omega, \quad f = 0 \text{ en } \partial\Omega.$$

Nuestro principal resultado en este caso es el siguiente: para cada $t_0 >$ fijo existe $C > 0$ tal que se mantiene la desigualdad siguiente

$$\left| (1+t)^{\frac{1}{m(p-1)-1}} u(x,t) - f(x) \right| \leq C f(x) (1+t)^{-1} \quad \text{para todo } t \geq t_0,\ x \in \Omega,$$

donde C depende solo de p, m, N, u_0, Ω y t_0.

La prueba utiliza las técnicas de reescalamiento y transformación en tiempo, comparación con sub y super-soluciones construidas en términos de las soluciones autosemejantes (también conocidas como soluciones de Barenblatt) de la (DNLE). Vamos a aplicar la estrategia de los artículos [8] y [104] para el caso $m > 1$ de la (PME), y resolveremos los problemas causados por la nolinealidad del operador p-Laplaciano.

(II) El caso cuasilineal $m(p-1) = 1$. Este caso es interesante para estudiar, dado que la ecuación hereda algunas características comunes de la Ecuación del Calor, $u_t = \Delta u$ (que recuperamos cuando $m = 1$ y $p = 2$): ambas ecuaciones son invariantes bajo multiplicación escalar, y se sabe que la solución general del problema de Dirichlet para la (DNLE) converge, después reescalamiento, a una función estacionaria cuya potencia m es solución del problema de valor propio para el operador p-Laplaciano. Sin embargo, cuando $(m, p) \neq (1, 2)$ aparecen diferencias a nivel de regularidad y comportamiento cualitativo. Mientras que las soluciones de la HE son C^∞, las soluciones de la (DNLE) tienen regularidad limitada debido a la doble degeneración de la ecuación en los niveles $u = 0$ y $\nabla u = 0$.

Estudiamos el comportamiento asintótico de las soluciones de la (DNLE) cuando $m(p-1) = 1$. Nuestro estudio utiliza el trabajo preliminar [86] y requiere una técnica delicada de barreras inspirada del artículo [26] sobre la estabilización de la solución en el caso de difusión rápida.

Para ser precisos, consideramos la transformación $v(x,t) = e^{\lambda_1 t} u(x,t)$, donde λ_1 es el primer valor propio del operador p-Laplaciano. La convergencia, a lo largo de subsucesiones de tiempo, de $v(x,t)$ a un posible perfil asintótico S se puede encontrar en el artículo [86] de Manfredi y Vespri. Los perfiles asintóticos elevados a potencia m están incluidos en el conjunto de soluciones no-negativas del problema elíptico correspondiente

$$-\Delta_p V = \lambda_1 V^{p-1} \text{ in } \Omega, \quad V = 0 \text{ on } \partial\Omega, \ V > 0 \text{ in } \Omega. \tag{1}$$

Se sabe que las soluciones del problema (1) forman un conjunto lineal, i.e., tienen la forma $\{cV_1 : c > 0\}$, con V_1 una solución particular normalizada (una p-función propia normalizada), cf. [6, 85].

En este trabajo completamos el análisis asintótico demostrando la convergencia uniforme para todo $t \to \infty$ de la solución rescalada $v(x,t)$ a un único perfil asintótico; también probamos una versión más fuerte en error relativo de esta convergencia. Nuestro resultado principal en el caso cuasilineal es el siguiente: existe una única constante $c > 0$ tal que

$$\lim_{t\to\infty} \left\| \frac{u(t,\cdot)}{\mathcal{U}(t,\cdot)} - 1 \right\|_{L^\infty(\Omega)} = \lim_{t\to\infty} \left\| \frac{v(t,\cdot)}{S(\cdot)} - 1 \right\|_{L^\infty(\Omega)} = 0,$$

donde $\mathcal{U}(x,t) = e^{-\lambda_1 t} S(x)$, $S(x) = cV_1^m$ con V_1 una solución que fijamos de la ecuación (1).

Las pruebas detalladas de los resultados mencionados serán dadas en el Capítulo 1. Los resultados han sido publicados en el artículo [98].

0.2.2 La ecuación de Fisher-KPP con difusión fraccionaria no lineal

El problema con difusión estándar proviene de los trabajos de Kolmogorov, Petrovskii y Piskunov (KPP), véase [77]. Esta es probablemente la ecuación más sencilla de reacción-difusión relacionada con la concentración u de una sustancia en una dimensión espacial

$$\partial_t u = D u_{xx} + f(u). \tag{2}$$

La opción $f(u) = u(1-u)$ produce la ecuación de Fisher [60] que fue originalmente utilizada para describir la propagación de una población biológica. Su resultado dice que el comportamiento para tiempo grandes de cualquier solución $u(x,t)$, con datos adecuados $0 \le u_0(x) \le 1$ que decaen rápidamente en el infinito, se parece a una onda viajera con una velocidad determinada [60, 77]. Al considerar la ecuación (2) en dimensiones $N \ge 1$, el problema se escribe

$$u_t - \Delta u = u(1-u) \quad \text{en } \mathbb{R}^N \times (0, +\infty). \tag{3}$$

Este caso ha sido estudiado en [10] por Aronson y Weinberger, donde extienden el caso unidimensional a dimensiones superiores. El resultado está formulado en términos de propagación lineal de los conjuntos de nivel de la solución. En el caso del modelo más general

$$u_t - \Delta u^m = u(1-u) \tag{4}$$

se tiene el mismo resultado de antes en el caso de difusión lenta $m > 1$ (Vázquez y de Pablo [50]).

A partir de estos resultados, King y McCabe examinaron en [75] el caso de difusión rápida $m < 1$ de la ecuación (4). Para $(N-2)_+/N < m < 1$ demostraron que el problema no admite soluciones de tipo onda viajera (TW) demostrando que los conjuntos de nivel de las soluciones del problema de valor inicial con datos iniciales adecuados se propagan en tiempo de manera exponencial.

Por otro lado e independientemente, Cabré y Roquejoffre ([37]) estudiaron el caso de difusión lineal fraccionaria $u_t(x,t) + (-\Delta)^s u(x,t) = f(u)$, donde $(-\Delta)^s$ es el operador Laplaciano fraccionario con $s \in (0,1)$ y llegaron a una conclusión similar: que no hay ningún comportamiento de tipo onda viajera cuando $t \to \infty$, y en efecto los conjuntos de nivel se propagan en tiempo de manera exponencial. Esto fue una sorpresa ya que su problema contiene difusión lineal.

Motivados por estos dos ejemplos de la rotura de la estructura asintótica de TW, estudiamos en [97] el caso de una difusión fraccionaria y no lineal. Más exactamente, consideramos el siguiente problema de reacción-difusión

$$\begin{cases} u_t(x,t) + (-\Delta)^s u^m(x,t) = f(u) & \text{para } x \in \mathbb{R}^N \text{ y } t > 0, \\ u(x,0) = u_0(x) & \text{para } x \in \mathbb{R}^N, \end{cases} \tag{KPP}$$

Estamos interesados en las propiedades de propagación de soluciones no-negativas y acotadas de este problema relacionando los resultados con la teoría de Fisher-KPP clásica. Suponemos que el término de reacción $f(u)$ satisface: $f \in C^1([0,1])$ es una función cóncava con $f(0) = f(1) = 0$, $f'(1) < 0 < f'(0)$. Por ejemplo podemos tomar $f(u) = u(1-u)$. El dato inicial $u_0(x) : \mathbb{R}^N \to [0,1]$ satisface una condición de crecimiento de la forma

$$0 \le u_0(x) \le C|x|^{-\lambda(N,s,m)}, \quad \forall x \in \mathbb{R}^N, \tag{5}$$

donde el exponente $\lambda(N,s,m)$ tiene una forma explícita que cambia con el exponente m. En nuestro trabajo demostramos que en todos los casos tenemos un ritmo exponencial de propagación de los conjuntos de nivel y concluimos que no hay ondas viajeras lineales como sucede cuando $s = 1$ and $m < 1$. La prueba se basa en un nuevo método de barrera que involucra sub y supersoluciones explícitas: la suposición (5) sobre u_0 es suficiente para demostrar el ritmo exponencial de la propagación de los conjuntos de nivel de la solución correspondiente u. Esto está basado en el hecho que las soluciones fundamentales (también llamadas soluciones de tipo Barenblatt) de la (FPME) $u_t(x,t) + (-\Delta)^s u^m(x,t) = 0$ tienen un comportamiento de tipo potencial cuando $|x| >> 1$, [107]. Recuérdese que debido a la nolinealidad en la parte de difusión, las soluciones no admiten una representación integral como sucede cuando $m = 1$.

La información esencial que necesitamos para calcular las tasas de expansión de los conjuntos de nivel es simplemente la velocidad de decaimiento de la solución fundamental cuando $|x| \to \infty$. Combinamos esta información con técnicas más habituales, como la linealización y comparación con sub y supersoluciones (argumento de barrera). También probamos estimaciones inferiores precisas para soluciones positivas de la (FPME), en la Sección 2.3, y hacemos un análisis adicional de las soluciones autosemejantes de la ecuación lineal con difusión fraccionaria en la Sección 2.4.

Este tema será tratado en el Capítulo 3. Los resultados han sido publicados en [97].

0.2.3 Ecuación de Medios Porosos con presión fraccionaria

El Capitulo 4 está dedicado al estudio del problema

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla P), \quad P = (-\Delta)^{-s}(u), \tag{PMFP}$$

para $m > 1$ y $u(x,t) \ge 0$. El problema se estudia para $x \in \mathbb{R}^N$, $N \ge 1$, y $t > 0$, con condiciones iniciales

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N,$$

donde $u_0 : \mathbb{R}^N \to [0, \infty)$ es acotada con soporte compacto ó decaimiento rápido en el infinito. La presión P se relaciona con la u a través de un operador potencial fraccionario lineal $P = \mathcal{K}(u)$. Para ser exactos, $\mathcal{K} = (-\Delta)^{-s}$ para $0 < s < 1$ con núcleo $K(x, y) = c|x-y|^{-(N-2s)}$ (es decir, un operador de Riesz). Esta ecuación ha sido primero introducida por Caffarelli y Vázquez en [39], cuando $m = 2$, como un modelo de difusión no lineal de tipo medios porosos con efectos de difusión no locales.

Cuando $m = 2$, en los artículos recientes [38–40] los autores han establecido las propiedades de velocidad de propagación finita, estimaciones a priori para las soluciones, la regularidad C^α , existencia de soluciones autosemejantes y comportamiento asintótico.

Un modelo similar fue propuesto por Head [67] para describir la dinámica de dislocación en cristales, vistos como un continuo: cuando $N = 1$, $s = 1/2$ la ecuación se convierte en (PMFP), es decir $\partial_t u = \nabla \cdot (u \nabla (-\Delta)^{-1/2}(u))$. Sea la densidad de dislocación u y definimos v mediante $u = v_x$; entonces v resuelve el “problema integrado” $v_t + |v_x|(-\partial_{xx})^{1-s} v = 0$. Esta última ecuación se ha estudiado recientemente por Biler, Karch y Monneau en [22], donde prueban la existencia de una solución de viscosidad única.

En los artículos [94, 95], establecemos resultados de existencia para cierta clase de soluciones débiles para $1 < m < 3$, para lo cual determinamos si la propiedad del soporte compacto se conserva en el tiempo o no, con dependencia explícita en el parámetro m. Este resultado está motivado por la velocidad de propagación finita que se da cuando $m = 2$. En efecto, descubrimos que el caso $m = 2$ es un caso límite: cuando $m \in [1, 2)$ el problema tiene velocidad de propagación infinita, mientras que para $m \in [2, \infty)$ tiene una velocidad de propagación finita. Esta es una característica típica del caso con difusión fraccionaria, $0 < s < 1$, de hecho cuando $s = 0$, la ecuación (CV) se convierte en la (PME) estándar $u_t = \Delta u^m$ para la cual la propiedad de velocidad de propagación finita es cierta para todo $m > 1$, cf. [106].

Una dificultad principal en este trabajo es la falta de unicidad y comparación de las soluciones, como se ha notado en [39]. La existencia se prueba por aproximación de problema (PMFP) a través de la regularización, eliminación de la degeneración y la reducción del dominio espacial. El problema aproximado puede resolverse deduciendo las estimaciones de energía adecuadas y luego pasando al límite mediante un argumento de compacidad parabólico. Para todo $m \geq 2$ se demuestra un comportamiento exponencial de tipo cola en el espacio. Esta es una información esencial para poder controlar las estimaciones de energía cuando $m < 3$. Desafortunadamente, cuando $m \geq 3$, el decaimiento exponencial no es suficiente para garantizar la compacidad necesaria en nuestro método. Esta es la razón por la cual nuestro resultado sobre existencia es verdad cuando $m < 3$, y lo que nos está llevando a atacar el problema de la existencia cuando $m \geq 3$ con técnicas diferentes y nuevas, actualmente bajo investigación.

Cuando $m \geq 2$, probamos la propiedad de velocidad de propagación finita para soluciones de (PMFP). Más precisamente, mostramos que los datos iniciales u_0 de soporte compacto producen soluciones $u(x,t)$ que tienen soporte compacto en espacio para todos $t > 0$. De hecho, podemos construir una función de explícita $U(x,t)$ con soporte compacto en el espacio, que representa una barrera superior para u. Encontrar tal U es una tarea que no es trivial debido a la falta de un principio de comparación; la prueba se basa en argumentos delicados de contradicción en el primer punto en espacio y tiempo donde $u(x,t)$ toca la barrera $U(x,t)$ por abajo. La prueba utiliza ideas del caso $m = 2$ tratado en [39].

La propagación infinita para $m \in (1,2)$, en dimensión $N = 1$, se realiza a través de un estudio de las propiedades del problema integrado $v_t = -|v_x|^{m-1}(-\Delta)^{1-s}v$, donde $v_x = u$. Este nuevo problema tiene la ventaja de cumplir un principio de comparación, una propiedad que no la teníamos para el modelo PMFP original. La herramienta principal es la construcción de una barrera inferior adecuada. Se utiliza una nueva forma especial de principio de comparación parabólico para el problema integrado, que se adapta al carácter no local del operador Laplaciano Fraccionario.

Transformación de soluciones. Es interesante encontrar una relación entre las soluciones de los dos modelos (FPME) y (PMFP), que ahora escribimos con notación diferente

$$u_t + (-\Delta)^s u^m = 0 \qquad \text{(FPME)}$$

$$\partial_t v = \nabla \cdot (v^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}v). \qquad \text{(PMFP)}$$

En [96] damos una fórmula de transformación explícita entre las soluciones autosemejantes de estos dos modelos en unos rangos especiales de parámetros $m \in (1,\infty) \longleftrightarrow \widetilde{m} \in (1,2)$, donde $\widetilde{s} = 1 - s$. Esto proporciona un resultado parcial de velocidad de propagación infinita para el (PMFP) en dimensiones mayores cuando $\widetilde{m} < 2$. Estos resultados serán demostrados en el Capítulo 5.

Trabajo en curso. En esta tesis doctoral probamos el resultado de existencia en el caso $m \in (2,3)$ como un límite de soluciones a problemas aproximados. La prueba de los resultados de existencia con $m \geq 3$ está aún bajo estudio y aparecerá en un próximo artículo en colaboración con J.L. Vázquez y F. del Teso, [95]. Los resultados relativos a la velocidad de propagación finita también funcionan para $m \geq 3$, como se indica a lo largo de las pruebas.

Introduction and summary of the results

The Heat Equation is the simplest equation describing diffusion processes:

$$u_t = \Delta u, \tag{HE}$$

where $u(x,t)$ denotes the concentration/density of the substance. This description was first stated by Fourier in 1822 in his celebrated book *Théorie Analytique de la Chaleur* [62]. The Heat Equation is well-known to be connected to the study of Browning motion of particles and it represents the basic model of linear diffusion, since the pioneering paper of Einstein [56].

There is a large class of physical systems that can not be described by linear PDEs. *Nonlinear partial differential equations* are the modern tool for describing many physical events in fluid dynamics, population dynamics, elasticity, relativity, thermodynamics. An important class of nonlinear PDEs are the parabolic second order partial differential equations used to describe diffusion processes of the above kind taking place under different conditions, making the model somehow more realistic.

The study of nonlinear partial differential equations is quite complex. In this thesis we will present three different models of nonlinear diffusion that require the use of different techniques. Usually, $u(x,t)$ denotes the density of a fluid or population at a time $t \geq 0$ and spatial coordinate $x \in \mathbb{R}^N$. It turns out that $u(x,t)$ solves a parabolic PDE as mentioned above. We are interested in the following problems: existence and uniqueness of solutions $u(x,t)$ and their asymptotic behaviour for large times. More specifically, the study of self-similar solutions and traveling wave solutions lead to a better understanding of the properties of u, such as convergence (with rates) to a stationary profile, positivity estimates or appearance of free boundaries.

0.3 Physical applications

0.3.1 Physical model

The basic model that we will consider was derived independently by Leibenzon and Muskat around 1930 from the consideration of the flow of an isentropic gas through a porous medium. An earlier application can be found in the study of groundwater infiltration by Boussisnesq in 1903.

Consider a continuum (fluid or population) represented by a density distribution $u(x,t) \geq 0$ that evolves with time following a velocity field $\mathbf{v}(x,t)$ according to the continuity equation

$$u_t = \nabla(u \cdot \mathbf{v}).$$

(i) Fluids in porous media behave according to Darcy's law: the velocity $\mathbf{v}$ derives from a potential

$$\mathbf{v} = -\nabla P, \tag{6}$$

in this case P denoting the pressure.

(ii) The relation between P and u: for gasses in porous media, Leibenzon [83] and Muskat [87] (1930) derived a relation in the form of the state law

$$P = f(u),$$

where f is a nondecreasing scalar function. Such function $f(u)$ is linear when the flow is isothermal and is a higher power of u when the flow is adiabatic, i.e. $f(u) = cu^{m-1}$ with $c > 0$ and $m > 1$.

The linear dependence $f(u) = cu$ was also obtained by Boussinesq [33] in 1903 when modelling water infiltration in an almost horizontal soil layer, in which case $u_t = c\Delta u^2$.

Therefore, the model can be written in the form $u_t = (c/m)\,\Delta u^m$, or, after rescaling the time variable,

$$u_t = \Delta u^m.$$

This equation is known as *The Porous Medium Equation*. It is the typical nonlinear diffusion model which stands as the basis for this dissertation. We refer to the book of Vázquez [106] for further details.

0.3.2 Nonlinear diffusion

Two of the most popular nonlinear diffusion models are the following:

(i) The Porous Medium Equation with $m > 0$

$$u_t = \Delta u^m. \tag{PME}$$

(ii) The p-Laplacian evolution equation, with $p > 1$,

$$u_t = \Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u). \tag{PLE}$$

The (PME) and (PLE) are both nonlinear equations of parabolic type belonging to the class

$$u_t = \nabla \cdot (A(u, \nabla u)\nabla u).$$

The (PME) is *density dependent diffusivity*: the diffusion coefficient $A(u) = mu^{m-1}$ makes the equation degenerate for $m > 1$ and singular for $m < 1$. The (PLE) belongs to the class of *gradient dependent diffusion equations*: $A(\nabla u) = |\nabla u|^{p-2}$ which degenerates for $p > 2$ and is singular for $p < 2$. In the particular case $m = 1$, respectively $p = 2$ we recover the classical *Heat Equation* (HE).

When $m < 1$ the (PME) is called *The Fast Diffusion Equation* with applications in plasma physics. Also the Fast Diffusion equation was obtained by John King [76] describing diffusion of impurities in silicon.

The terminology *slow/fast diffusion* refers to the property of finite speed of propagation which depends of the range $m > 1$ and $m < 1$ for the PME, respectively $p > 2$ and $1 < p < 2$ for the PLE. In [94] we investigate this property in the case of the more general model (PMFP).

The $p-$Laplacian equation (PLE) is used to describe the filtration through a porous medium for a non-newtonian fluid, see the book Ladyzhenskaya [81] on viscous incompressible flows.

Diffusion equations have a wide applicability in image processing. For instance, we mention: image denoising based on anisotropic diffusion equations (Perona-Malik model [88]), application of the fast PLE (with $p < 2$) by Barenblatt and Vázquez for image contour enhancement [14]. See also Barbu [12], Weickert [113], Catté et all [44].

By combining both types of nonlinearities, we obtain the Doubly Nonlinear Diffusion Equation

$$u_t = \Delta_p u^m \tag{DNLE}$$

with $m > 0$, $p > 1$. This general model inherits the properties of the (PME) and (PLE).

These models have been intensively studied; we mention various works of Vázquez: the monograph [106] for the theory of the (PME), the book [105] for estimates and scaling, the survey [104] for the asymptotic behaviour. The theory of parabolic equations started with the work of Ladyzhenskaya, Solonnikov and Ural'seva [80]. More recently, we mention the book of DiBenedetto [55].

0.3.3 Nonlocal diffusion

Diffusion processes with long-range effects are modeled in many situations using nonlocal operators. Of special interest in this thesis is the Fractional Laplacian operator (Stein [99], Landkof [82]), usually defined via the Fourier Transform

$$\mathcal{F}((-\Delta)^s f(x)) = |\xi|^{2s}\mathcal{F}(f)(\xi),$$

for functions f is the Schwartz class. When $0 < s < 1$, the Fractional Laplacian operator can also be defined by the integral formula using hyper-singular kernels

$$(-\Delta)^s f(x) = C_{N,s}\,\text{P.V.}\int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^{N+2s}}dy,$$

where P.V. stands for the principal value and $C_{N,s} = \pi^{-(2s+N/2)}\Gamma(N/2+s)/\Gamma(-s)$ is a normalization constant. When $s \to 0$ we recover the identity operator, while for $s \to 1$ we recover the standard Laplacian. The inverse operator is given by the Riesz Potential

$$(-\Delta)^{-s} f(x) = C_{N,-s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}} dy.$$

The *Fractional Heat Equation*

$$u_t + (-\Delta)^s u = 0.$$

models anomalous diffusion present in a number of phenomena in physics, finance, biology, [...].

In probability, the fractional Laplacian operator is the infinitesimal generator of stable Lévy processes [7, 18, 100]. The use of fractional operators in modelling diffusion processes has been known recently by the work of Caffarelli, Vázquez, Athanasopoulos, Salsa, Silvestre, Kassmann, Cabré, Valdinoci and many others, most of the topics on stationary problems.

Anomalous diffusion is often described by nonlinear models with fractional operators. We present now two models for flows in porous media involving nonlocal diffusion effects.

(i) Nonlocal PME. The first model presented is the Fractional Porous Medium Equation, $m > 1$, $s \in (0,1)$

$$u_t + (-\Delta)^s u^m = 0 \qquad \text{(FPME)}$$

where $(-\Delta)^s$ is the Fractional Laplacian Operator described above. This model can be interpreted as the nonlocal corespondent of the classical PME ($s = 1,\ m > 1$). Major differences appear at the level of propagation: on one hand the PME has finite speed of propagation, which is usually expressed by showing that compactly supported initial data produce compactly supported solutions (therefore, there appear free boundaries). On the other hand, its nonlocal correspondent, the (FPME) has infinite speed of propagation, i.e. even if u_0 is compactly supported, the solution is strictly positive for all times $u(x,t) > 0$, $t > 0$. This latter property is a consequence of the nonlocal character of the operator when $s \in (0,1)$. The (FPME) has been studied by Vázquez and collaborators [28, 46–48] and many others.

Reaction-diffusion equations appear when modelling diffusion combined with reaction phenomena. When the reaction term is of the form $u(1-u)$, the model is known as *Fisher-KPP* and was first used to describe the spreading of biological population by Fisher [60] and Kolmogorov, Petrovskii and Piskunov [77]. In the following Section 0.4.2 we describe a model of nonlinear fractional diffusion combined with Fisher-KPP type reaction $u_t + (-\Delta)^s u^m = u(1-u)$.

(ii) PME with fractional pressure. Consider the case when the pressure P is related to u by a nonlocal operator of fractional type

$$P = \mathcal{K}(u)$$

where $\mathcal{K} = (-\Delta)^{-s}$ is the inverse of the fractional Laplacian operator, $0 < s < 1$. The diffusion model with nonlocal effects is the following

$$u_t = \nabla \cdot (u \nabla P), \quad P = (-\Delta)^{-s}(u). \tag{CV}$$

This has been introduced by Caffarelli and Vázquez [39], then its properties have been studied in a series of papers [38, 40]. On the other hand, a similar model was proposed by Head [67] to describe the dynamics of dislocation in crystals seen as a continuum [67]. In the limit $s \to 1$ (Serfaty and Vázquez in [91]), it results a "mean field" equation arising in superconductivity and superfluidity.

Departing from equation (CV), more general models can be considered that maintain the main features of the PME. We discuss here the following problem

$$u_t = \nabla \cdot (u^{m-1} \nabla P), \quad P = (-\Delta)^{-s}(u). \tag{PMFP}$$

We call this problem *Porous Medium with Fractional Pressure*, briefly (PMFP). The nonlinearity has surprising effects on the propagation property of the solution, as we will describe in Section 0.4.3.

0.4 Main Results

We now present the results obtained by the author in the preparation of this Dissertation Memoir.

0.4.1 The Doubly Nonlinear Diffusion equation

We consider the problem

$$u_t = \Delta_p u^m \tag{DNLE}$$

where $m > 0$, $p > 1$ and $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is the $p-$Laplacian operator.

When $p = 2$ we recover the porous medium equation $u_t = \Delta u^m$ with $m > 1$ while, when $m = 1$, we recover the degenerate p-Laplacian equation (PLE) $u_t = \Delta_p u$ with $p > 2$, both well known equations in the literature.

As references for the previous theory for the (DNLE) we mention: the asymptotic behaviour of the Cauchy Problem by Carrillo, Agueh, Blanchet [2], existence and uniqueness of the Dirichlet problem in a bounded domain by Manfredi, Vespri, Savaré [86, 90] estimates by Bonforte, Grillo [30], regularity by Vespri, Porzio [89], Kuusi, Urbano [79].

For the regularity of the $p-$Laplacian equation we mention the work of Kuusi and Mingione [78]. See also Bonforte, Di Castro [29] on quantitative local estimates for nonlinear elliptic equations involving p-Laplacian type operators. Recent works on regularity in the singular case by Fornaro, Sosio and Vespri [61].

We investigate the homogenous Dirichlet problem in a bounded domain $\Omega \subset \mathbb{R}^N$ with initial data $u(x,0) = u_0(x)$ in the range of parameters $m(p-1) \geq 1$. For non-negative and integrable data u_0, we study the asymptotic behaviour of the positive solution $u(x,t)$ of Problem (DNLE). Many of our results are new even in the p-Laplacian case $m = 1$, $p > 2$.

(I) The degenerate case $m(p-1) > 1$. We proved sharp rates of convergence of the rescaled solution $v(x,\tau) = t^{\frac{1}{m(p-1)-1}} u(x,t)$, $t = e^\tau$ to its unique asymptotic profile $f(x)$, as $\tau \to +\infty$. The stationary profile f can be characterized as the positive solution to the corresponding stationary problem

$$\Delta_p f^m + \frac{1}{m(p-1)-1} f = 0 \text{ in } \Omega, \quad f = 0 \text{ on } \partial\Omega.$$

Our main result in this case is the following: for every $t_0 >$ fixed there exists $C > 0$ such that the following inequality holds

$$\left| (1+t)^{\frac{1}{m(p-1)-1}} u(x,t) - f(x) \right| \leq C f(x)(1+t)^{-1} \quad \text{for all } t \geq t_0 \text{ and } x \in \Omega,$$

where C depends only on p, m, N, u_0, Ω and t_0.

The proof uses the techniques of rescaling and time transformation, comparison with sub and super-solutions constructed in terms of the self-similar solutions (also known as Barenblatt solutions) of the (DNLE). We applied the strategy of the papers [8] and [104] for the case $m > 1$ of the PME, and we have solved the difficult problems caused by the nonlinearity of the p-Laplacian operator.

(II) The quasilinear case $m(p-1) = 1$. This case is interesting to study since the equation inherits some common features of the Heat Equation, $u_t = \Delta u$ (which can be recovered when $m = 1$ and $p = 2$): this equation is invariant under scalar multiplication, and it is known that a general solution converges after rescaling to one of the (stationary) solutions of the eigenvalue problem for the $p-$Laplacian operator. However, when $(m,p) \neq (1,2)$ differences appear at the level of regularity and qualitative behaviour. While solutions of the HE are C^∞ smooth, solutions of the (DNLE) have limited regularity due to the double degeneracy of the equations at the levels $u = 0$ and $\nabla u = 0$.

We study the asymptotic behaviour of solutions of the (DNLE) when $m(p-1) = 1$. Our study uses the preliminary work [86] and requires a delicate barrier technique inspired from the work [26] on fast diffusion stabilization.

To be precise, we consider the rescaling $v(x,t) = e^{\lambda_1 t} u(x,t)$, where λ_1 is the first eigenvalue of the p-Laplacian operator Δ_p. The convergence, along time subsequences, of $v(x,t)$ to a possible asymptotic profile S can be found in the paper [86] of Manfredi and Vespri. The possible asymptotic profiles S, when taken to power m (that is S^m), are included in the set of non-negative solutions of the corresponding elliptic problem

$$-\Delta_p V = \lambda_1 V^{p-1} \text{ in } \Omega, \quad V = 0 \text{ on } \partial\Omega,\ V > 0 \text{ in } \Omega. \tag{7}$$

It is known that the set of solutions to Problem (7) is a linear set, i.e., they have the form $\{cV_1 : c > 0\}$, where V_1 is a particular normalized solution (a normalized p-eigenfunction), cf. [6, 85].

In this work, we complete the asymptotic analysis by proving uniform convergence of the rescaled solution $v(x,t)$ to an unique asymptotic profile; this happens for all times $t \to \infty$ and we also prove a stronger relative error version of this convergence. Our main result in the quasilinear case is the following: there exists a unique constant $c > 0$ such that

$$\lim_{t\to\infty} \left\| \frac{u(t,\cdot)}{\mathcal{U}(t,\cdot)} - 1 \right\|_{L^\infty(\Omega)} = \lim_{t\to\infty} \left\| \frac{v(t,\cdot)}{S(\cdot)} - 1 \right\|_{L^\infty(\Omega)} = 0,$$

where $\mathcal{U}(x,t) = e^{-\lambda_1 t} S(x)$, $S(x) = cV_1^m$ where V_1 is a fixed solution of (1.8).

The detailed proof of the results mentioned above will be given in Chapter 1. We refer to [98].

0.4.2 The Fisher-KPP equation with nonlinear fractional diffusion

The problem with standard diffusion goes back to the work of Kolmogorov, Petrovskii and Piskunov, see [77]. This is probably the most simple reaction-diffusion equation concerning the concentration u of a single substance in one spatial dimension,

$$\partial_t u = D u_{xx} + f(u). \tag{8}$$

The choice $f(u) = u(1-u)$ yields Fisher's equation [60], that was originally used to describe the spreading of biological populations. Their celebrated result says that the long-time behaviour of any solution of $u(x,t)$, with suitable data $0 \le u_0(x) \le 1$ that decay fast at infinity, resembles a traveling wave with a definite speed, [60, 77]. When considering equation (8) in dimensions $N \ge 1$, the problem becomes

$$u_t - \Delta u = u(1-u) \quad \text{in } \mathbb{R}^N \times (0, +\infty).$$

This case has been studied in [10] by Aronson and Weinberger, where they extend the one-dimensional case to higher dimensions. The result is formulated in terms of linear propagation of the level sets of the solution. In the case of the more general model

$$u_t - \Delta u^m = u(1-u) \tag{9}$$

the same result as before holds in the case of slow diffusion $m > 1$ (Vázquez and de Pablo [50]).

Departing from these results, King and McCabe examined in [75] the case of fast diffusion $m < 1$ of equation (9). For $(N-2)_+/N < m < 1$, they showed that the problem does not admit traveling wave (TW) solutions by proving that level sets of the solutions of the initial-value problem with suitable initial data propagate exponentially fast in time.

On the other hand, and independently, Cabré and Roquejoffre ([37]) studied the case of fractional linear diffusion $u_t(x,t)+(-\Delta)^s u(x,t) = f(u)$, where $(-\Delta)^s$ is the Fractional Laplacian operator with $s \in (0,1)$ and they concluded in the same vein that there is no traveling wave behaviour as $t \to \infty$, and indeed the level sets propagate exponentially fast in time. This came as a surprise since their problem deals with linear diffusion.

Motivated by these two examples of break of the asymptotic TW structure, we studied in [97] the case of a diffusion that is both fractional and nonlinear. More exactly, we consider the following reaction-diffusion problem

$$\begin{cases} u_t(x,t) + (-\Delta)^s u^m(x,t) = f(u) & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{KPP}$$

We are interested in the propagation properties of nonnegative and bounded solutions of this problem in the spirit of the Fisher-KPP theory. We assume that the reaction term $f(u)$ satisfies $f \in C^1([0,1])$ is a concave function with $f(0) = f(1) = 0, \quad f'(1) < 0 < f'(0)$. For example we can take $f(u) = u(1-u)$. The initial datum $u_0(x) : \mathbb{R}^N \to [0,1]$ and satisfies a growth condition of the form

$$0 \le u_0(x) \le C|x|^{-\lambda(N,s,m)}, \quad \forall x \in \mathbb{R}^N, \tag{10}$$

where the exponent $\lambda(N,s,m)$ has an explicit form which changes with the exponent m. In our work we prove that in all cases we have an exponential rate of propagation of level sets and we conclude that there are no linear traveling waves, as it happens when $s = 1$ and $m < 1$. Our proof is based on a new barrier method that involves explicit sub and supersolutions: assumption (10) on u_0 is sufficient to prove the exponential rate of propagation of the level sets of the corresponding solution $u(x,t)$. This is based on the fact that the fundamental solution (also called Barenblatt solution) of the (FPME) $u_t(x,t) + (-\Delta)^s u^m(x,t) = 0$ has a power-like tail-behaviour when $|x| >> 1$, cf. [107]. Recall that due to the nonlinearity in the diffusion part, solutions do not admit an integral representation as it happens when $m = 1$.

The essential information that we need to calculate the expansion rates of the level sets is merely the decay rate of the tail of Barenblatt solution as $|x| \to \infty$. We combine this information with more standard techniques, such as linearization and comparison with sub- and super-solutions (barrier argument). We also prove accurate lower estimates for positive solutions of the (FPME), cf. Section 2.3, and we provide a further selfsimilar analysis of the linear diffusion, cf. Section 2.4.

This is contained in Chapter 3. The results have been published in [97].

0.4.3 Porous Medium Equation with fractional pressure

Chapter 4 is devoted to the study of the problem

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla P), \quad P = (-\Delta)^{-s}(u), \tag{PMFP}$$

for $m > 1$ and $u(x,t) \geq 0$. The problem is posed for $x \in \mathbb{R}^N$, $N \geq 1$, and $t > 0$, and we give initial conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N,$$

where $u_0 : \mathbb{R}^N \to [0,\infty)$ is bounded with compact support or fast decay at infinity. The pressure P is related to u through a linear fractional potential operator $P = \mathcal{K}(u)$. To be specific $\mathcal{K} = (-\Delta)^{-s}$ for $0 < s < 1$ with kernel $K(x,y) = c|x-y|^{-(N-2s)}$ (i.e. a Riesz operator). This equation has been first introduced by Caffarelli and Vázquez in [39], when $m = 2$, as a model for nonlinear diffusion of porous medium type with nonlocal diffusion effects.

When $m = 2$, in the recent papers [38–40] the authors have established the properties of finite speed of propagation, a priori estimates for the solutions, C^α regularity, existence of self-similar solutions and asymptotic behaviour.

A similar model was proposed by Head [67] to describe the dynamics of dislocations in crystals seen as a continuum: when $N = 1$ the equation becomes the (PMFP), namely $\partial_t u = \nabla \cdot (u \nabla (-\Delta)^{-1/2}(u))$. Letting the dislocation density u, we define v by $u = v_x$. Then it occurs that v solves an "integrated problem" $v_t + |v_x|(-\partial_{xx})^{1-s} v = 0$. This latter equation has been recently studied by Biler, Karch and Monneau in [22], where they prove existence of a unique viscosity solution.

In the papers [94, 95] we establish existence results for a certain class of weak solutions for $1 < m < 3$, for which we determine whether the property of compact support is conserved in time or not, with explicit dependence on the parameter m. This result is motivated by the finite speed of propagation that happens for $m = 2$. Indeed, we discover that the case $m = 2$ is a borderline case: when $m \in [1, 2)$ the problem has infinite speed of propagation, while for $m \in [2, \infty)$ it has finite speed of propagation. This is a feature typical of the fractional diffusion case, $0 < s < 1$, indeed when $s = 0$, equation (CV) becomes the standard PME $u_t = \Delta u^m$ for which the property of finite speed of propagation holds for all $m > 1$, cf. [106].

A main difficulty in this work is the lack of uniqueness and comparison of the solutions, as already noticed in [39]. The existence is proved by approximating problem (PMFP) through regularization, elimination of the degeneracy and reduction of the spatial domain. The approximated problem can be solved by deriving suitable energy estimates and then passing to the limit using a parabolic compactness argument. For all $m \geq 2$ we prove an exponential tail behaviour in space. This is an essential information to be able to control the energy terms when $m < 3$. Unfortunately, when $m \geq 3$, the exponential tail behaviour is not enough to guarantee the compactness needed in our method. This is the reason why our existence result holds when $m < 3$, and what is leading us to attack the existence problem for $m \geq 3$ with different and new techniques, currently under investigation.

When $m \geq 2$, we prove the property of finite speed of propagation for solutions to (PMFP). More precisely, we show that compactly supported initial data u_0 produce solutions $u(x,t)$ which have compact support in space for all $t > 0$. Indeed, we can construct an explicit function $U(x,t)$ with compact support in space, which represents an upper barrier for u. Finding such U is a task which is highly not trivial in view of the lack of comparison principle; the proof is based on delicate contradiction arguments at the first point in space and time where $u(x,t)$ touches the barrier $U(x,t)$ from below. The proof uses ideas from the case $m = 2$ treated in [39].

The infinite propagation for $m \in (1,2)$, in dimension $N = 1$, is done via a study of the properties of the integrated problem $v_t = -|v_x|^{m-1}(-\Delta)^{1-s}v$, where $v_x = u$. This new problems has the advantage of a comparison principle, a property that we did not have in the original PMFP model. The main tool is the construction of a suitable lower barrier and using a new special form of parabolic comparison principle for the integrated problem, which adapts to the nonlocal character of the fractional Laplacian operator.

Transformations of solutions. It is interesting to find a relation between solutions to the (FPME) and the (PMFP) models, that we rewrite with different notation

$$u_t + (-\Delta)^s u^m = 0 \qquad \text{(FPME)}$$

$$\partial_t v = \nabla \cdot (v^{\widetilde{m}-1} \nabla (-\Delta)^{-\widetilde{s}} v). \qquad \text{(PMFP)}$$

In [96] we give an explicit transformation formula between self-similar solutions of these two models in a special ranges of parameters $m \in (1,\infty) \longleftrightarrow \widetilde{m} \in (1,2)$, where $\widetilde{s} = 1-s$. This is a partial result of infinite propagation for the (PMFP) in higher dimensions when $\widetilde{m} < 2$. These results will be described in Chapter 5.

Work in Progress. In this dissertation we prove the existence result as a limit of solutions to the approximate problems in case $m \in (2,3)$. The proof of the existence result for $m \geq 3$ is still under study now and will appear in a forthcoming paper in collaboration with J.L. Vázquez and F. del Teso, [95]. The results concerning the finite propagation work also for $m \geq 3$, as stated throughout the proofs.

Chapter 1

The Doubly Nonlinear Diffusion Equation

This chapter is devoted to the study of the Dirichlet problem for the doubly nonlinear diffusion equation $u_t = \Delta_p u^m$, where $p > 1$, $m > 0$, posed in a bounded domain in $\mathbb{R}^N$ with homogeneous boundary conditions and with non-negative and integrable initial data. We consider the degenerate case $m(p-1) > 1$ and the quasilinear case $m(p-1) = 1$. In the first case we establish the large-time behaviour by proving the uniform convergence to a unique asymptotic profile and we also give rates for this convergence. In the second case the asymptotic profile is unique only up to a constant factor that we have to determine.

The results presented here are part of the paper [98].

1.1 Introduction

We are interested in describing the behaviour of non-negative solutions of the homogenous Dirichlet problem for the doubly nonlinear equation (DNLE) for large times. To be precise, we consider the following initial and boundary value problem

$$\begin{cases} u_t(x,t) = \Delta_p u^m(x,t) & \text{for } t > 0 \text{ and } x \in \Omega, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } t > 0 \text{ and } x \in \partial\Omega, \end{cases} \tag{DNLE-d}$$

for $m > 0$, $p > 1$. The problem is posed in a bounded domain $\Omega \in \mathbb{R}^N$ with initial data $u_0 \geq 0$, $u_0 \in L^1(\Omega)$ so that the solution $u(x,t) \geq 0$ too. The p-Laplacian operator is well-known to be defined as $\Delta_p w := \text{div}(|\nabla w|^{p-2}\nabla w)$. We study the large time asymptotic behaviour of solutions to Problem (DNLE-d) in the "degenerate case" $m(p-1) > 1$, also known as slow diffusion case, and in the "quasilinear case" $m(p-1) = 1$.

Let us first make some comments concerning the range $m(p-1) > 1$. When $p = 2$ we recover the porous medium equation (PME) $u_t = \Delta u^m$ with $m > 1$ while, when $m = 1$, we recover the degenerate p-Laplacian equation (PLE) $u_t = \Delta_p u$ with $p > 2$, both well known equations in the literature. Notice that in this paper we only require $m(p-1) > 1$, that also includes cases where either $m \le 1$ or $p \le 2$. The (PLE) and the (PME), as prototypes for degenerate diffusion, enjoy many common properties, such as finite speed of propagation and the existence of some special (self-similar) solutions, which play an important role in describing the asymptotic behaviour for general initial data. We refer to the Ph.D. thesis of R. Iagar [69] for a detailed characterization of the properties of these two nonlinear diffusion models.

In this paper we complete the panorama by analyzing in detail the large-time properties of the degenerate (DNLE), which combines the difficulties of both equations and offers some new challenges.

The quasilinear case $m(p-1) = 1$ is also interesting to study since it inherits some common features of the Heat Equation, $u_t = \Delta u$ (which can be recovered when $m = 1$ and $p = 2$): both equations are invariant under scalar multiplication, and it is known that a general solution of the (DNLE-d) converges after rescaling to one of the (stationary) solutions of the eigenvalue problem for the $p-$Laplacian operator. However, when $(m, p) \neq (1, 2)$ differences appear at the level of regularity and qualitative behaviour. While solutions of the HE are C^∞ smooth, solutions of the (DNLE) have limited regularity due to the degenerate (singular) parabolic character of the equations at the levels $u = 0$ and $\nabla u = 0$ (see Fig. 1.1).

The remaining "fast diffusion case" $m(p-1) < 1$ has quite different properties and deserves a separate study. Indeed, we deal in this case with singular diffusions, and new phenomena appear such as extinction in finite time, or lack of uniqueness of the asymptotic profile. All this gives a different flavor to the analysis of the asymptotic behaviour.

As references for the previous theory for the (DNLE) we mention [86] for the degenerate and quasilinear cases and [90] for the singular case. We mention also that the asymptotic behaviour of the Cauchy problem on $\mathbb{R}^N$ has been studied in [2]. Many of our results are new even in the p-Laplacian case $m = 1$, $p > 2$. We also remark that most of the techniques needed to prove existence, uniqueness and other basic properties of the parabolic (DNLE) flow can be taken from the books [105, 106] for the (PME), and [55] for the (PLE). We also refer to [8] and [104] for a complete asymptotic analysis of the Dirichlet problem on bounded domains, for the (PME) when $m > 1$. Also, we refer to Lindqvist [84] for a summary of $p-$Laplacian equations.

Presentation of the main results. The purpose of this work is to analyze completely the asymptotic behaviour of the (DNLE-d) on Euclidean bounded domains. For convenience we assume that the boundary $\partial\Omega$ is $C^{2,\alpha}$ smooth. Since the cases $m(p-1) > 1$ and $m(p-1) = 1$ involve different techniques, we will present them separately.

Ia. The degenerate case $m(p-1) > 1$**.** This work generalizes the asymptotic analysis carried out in the above mentioned papers [8, 104]. The outline of the theory is similar but the double nonlinearity asks for a number of interesting techniques. Throughout the

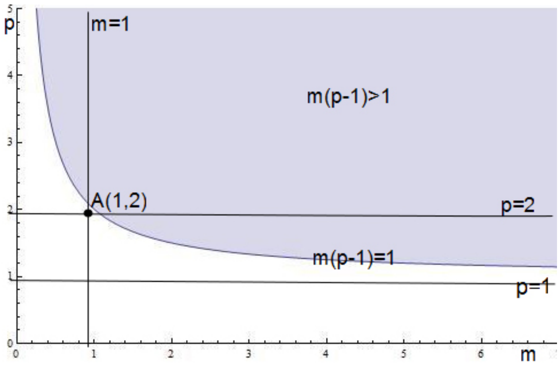


FIGURE 1.1: Ranges of parameters m and p

study we will fix the notation $\mu = 1/(m(p-1)-1) > 0$, since this quantity will appear frequently.

The asymptotic behaviour is better understood via the well-known method of rescaling and time transformation; let us introduce

$$v(x,\tau) = t^{\mu} u(x,t), \quad t = e^{\tau}. \tag{1.1}$$

In this way, the Dirichlet problem DNLE-d is transformed into

$$\begin{cases} v_\tau(x,\tau) = \Delta_p v^m(x,\tau) + \mu\, v(x,\tau) & \text{for } \tau \in \mathbb{R} \text{ and } x \in \Omega, \\ v(x,\tau) = 0 & \text{for } \tau \in \mathbb{R} \text{ and } x \in \partial\Omega, \\ v(x,0) = v_0 & \text{for } x \in \Omega. \end{cases} \tag{1.2}$$

In Section 1.2 we prove Theorem 1.2.1, which shows uniform convergence of the rescaled solution $v(x,\tau)$ to its unique asymptotic profile $f(x)$, as $\tau \to +\infty$. The stationary profile f can be characterized as the positive solution to the corresponding stationary problem

$$\Delta_p f^m + \mu\, f = 0 \text{ in } \Omega, \quad f = 0 \text{ on } \partial\Omega.$$

The result of this Theorem is not surprising, but it does not appear explicitly in literature and it is needed to prove the next results. The techniques used in this step follow the work [104] for the PME.

In Section 1.3 we prove sharp rates of convergence of $v(\tau) \to f$ as $\tau \to \infty$; this represents the first important result of this paper. Throughout we paper we denote $v(\tau)$ as a function of $\tau \ge 0$ with values in a function space: $v(\tau) : \Omega \to \mathbb{R}$, $v(\tau)(x) = v(x,\tau)$.

Theorem 1.1.1. ***(Weighted rate of convergence)*** *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$. Let $u(x,t)$ be the weak solution to Problem* (DNLE-d) *corresponding to a nonnegative initial datum $u_0 \in L^1(\Omega)$. Then for every $t_0 >$ fixed there exists $C > 0$ such that the following inequality holds*

$$|(1+t)^{\mu} u(x,t) - f(x)| \le C f(x)(1+t)^{-1} \quad \textit{for all } t \ge t_0 \textit{ and } x \in \Omega, \tag{1.3}$$

where C depends only on p, m, N, u_0, Ω and t_0.

In other words

$$u(x,t) = t^{-\mu} f(x) + O(f(x)\, t^{-\mu-1}). \tag{1.4}$$

Remarks. (i) *Sharpness.* Estimate (1.3) is sharp, since it is satisfied with equality when we consider the separate variable solution,

$$U(x,t;s) = (s+t)^{\mu} f(x), \tag{1.5}$$

with parameter $s > 0$.

(ii) *Convergence in relative error.* Let $U(x,t) := U(x,t;0)$ be the separate variable solution (1.5) and let

$$v(x,\tau) = t^{\frac{1}{m(p-1)-1}} u(x,t), \qquad t = e^{\tau}$$

be the *rescaled solution* given in (1.21). We can rewrite (1.39) in the following form:

Corolary 1.1.1. ***(Convergence in relative error)*** *Under the assumptions of Theorem* 1.1.1*, if u denotes the solution of Problem* (DNLE-d)*, then*

$$\left\| \frac{u(x,t)}{U(x,t)} - 1 \right\|_{L^\infty(\Omega)} = \left\| \frac{v(x,\tau)}{f(x)} - 1 \right\|_{L^\infty(\Omega)} \leq \mathcal{C} t^{-1}. \tag{1.6}$$

Here the $L^\infty(\Omega)$ norm is considered in the space variable x. The main idea will be to compare the general solution u of Problem (DNLE-d) with functions belonging to special families, more exactly self-similar solutions of the equation (DNLE) $u_t = \Delta_p u^m$. We will try to follow the strategy used in the papers [8] and [104] for the case $m > 1$ of the PME, and solve the problems caused by the nonlinearity of the p-Laplacian operator.

Ib. Selfsimilar study. In the process of proving the above results we became interested in the existence and properties of self-similar solutions of the equation $u_t = \Delta_p u^m$ (DNLE), that is, functions of the form

$$\mathcal{U}(x,t) = (t+s)^{-\alpha} h(r), \quad r = |x|(t+s)^{-\beta},$$

where α, β are positive parameters and h is a real valued function satisfying a certain ODE. As a by-product we give a formal characterization of such solutions. Selfsimilar solutions are often used as a way of indicating the behavior of a general solution of the (DNLE).

II. The quasilinear case. In Section 1.5 we study the asymptotic behaviour of solutions of the Dirichelt problem (DNLE-d) when $m(p-1) = 1$. Our study uses the preliminary work [86] and requires a delicate barrier technique inspired from the work

[26] on fast diffusion stabilization. To be precise, we consider the rescaling

$$v(x,t) = e^{\lambda_1 t} u(x,t), \tag{1.7}$$

where λ_1 is the first eigenvalue of the p-Laplacian operator Δ_p, [6, 85]. Notice that here there is no time transformation, but only rescaling.

This problem was previously studied by Manfredi and Vespri in [86], where the authors obtained the convergence, along time subsequences, of $v(x,t)$ to a possible asymptotic profile. At the same time they proved that the set of asymptotic profiles is included in the set of solutions of the corresponding elliptic problem

$$-\Delta_p V = \lambda_1 V^{p-1} \text{ in } \Omega, \quad V = 0 \text{ on } \partial\Omega,\ V > 0 \text{ in } \Omega. \tag{1.8}$$

It is known that when $\lambda = \lambda_1$ then the set of solutions is a linear set, i.e., they have the form $\{cV_1 : c > 0\}$, where V_1 is a particular normalized solution (a normalized p-eigenfunction), cf. [6, 85].

In this work we complete the asymptotic analysis by proving uniform convergence of the rescaled solution $v(x,t)$ to an unique asymptotic profile; this happens for all times $t \to \infty$ and we also prove a relative error version for this convergence. Our main result in the quasilinear case is the following.

Theorem 1.1.2. *Consider $m(p-1) = 1$. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded connected domain of class $C^{2,\alpha}$, $\alpha > 0$. Let $u(t,\cdot)$ be a weak solution to the Dirichlet Problem* (DNLE-d) *corresponding to the nonnegative initial datum $u_0 \in L^1(\Omega)$. Let $v(x,t) = e^{\lambda_1 t} u(x,t)$. Then there exists a unique constant $c > 0$ such that*

$$\lim_{t\to\infty} \left\| \frac{u(x,t)}{\mathcal{U}(x,t)} - 1 \right\|_{L^\infty(\Omega)} = \lim_{t\to\infty} \left\| \frac{v(x,t)}{S(x)} - 1 \right\|_{L^\infty(\Omega)} = 0, \tag{1.9}$$

where $\mathcal{U}(x,t) = e^{-\lambda_1 t} S(x)$ and $S^m = cV_1$.

In order to clarify the result, let us point out that the result states that there is a unique asymptotic profile of the form $S = V^{1/m}$, where V is one of the positive solutions of (1.8). In other words, there is a unique $c = c(u_0) > 0$ such that $V = cV_1$. Though the asymptotic constant c depends on the data, there is no explicit or semi-explicit formula to compute it. This is a typical occurrence issue of nonlinear evolution problems, see a discussion of the issue in [74] when studying the Barenblatt equation for elastoplastic filtration, a quite different model of nonlinear heat flow. In order to prove Theorem 1.1.2, the methods used in the degenerate case do not work anymore and therefore we apply a different method, a barrier argument, based on the one used in [26] to prove convergence in relative error for the fast diffusion equation.

1.2 Asymptotic behaviour for $m(p-1) > 1$

1.2.1 Preliminaries

In order to present the asymptotic behaviour, we first need to introduce some preliminary results concerning the smoothing effects of the (DNLE).

NOTATIONS.

$$Q_T = \Omega \times (0,T),\ Q = \Omega \times (0,\infty),\ d(x) = \mathrm{dist}(x, \partial\Omega).$$

The notion of weak solution is defined in the standard sense, we refer to [55]. In addition, it is known by standard semigroup theory for m-accretive operators that for every initial datum u_0 in $L^1(\Omega)$ there exists a unique non-negative weak solution $u \in C([0,\infty) : L^1(\Omega))$ of Problem (DNLE-d) with a number of regularity properties that we will mention as needed, for instance it satisfies the Maximum Principle.

Now, we illustrate specific properties concerning the smoothing effects of the (DNLE), properties that will be needed in our proofs (we refer for example to [105]). In what follows we assume that $\Omega \subseteq \mathbb{R}^N$ is a $C^{2,\alpha}$ domain, for some $\alpha \in (0,1)$. Let $u(x,t)$ be a weak solution to Problem (DNLE-d) corresponding to the nonnegative initial datum $u_0 \in L^1(\Omega)$.

1. Bénilan-Crandall type estimates

1. If $m(p-1) > 1$, then
$$u_t \geq -\mu t^{-1} u \tag{1.10}$$
in the sense of distributions.

2. If $m(p-1) < 1$, then
$$u_t \leq \mu t^{-1} u \tag{1.11}$$
in the sense of distributions.

Also, in the case $m(p-1) > 1$, the weak solution u of Problem (DNLE-d) verifies the following estimate
$$\|u_t(x, t+s)\|_1 \leq \mu (t+s)^{-1} \|u(s)\|_1. \tag{1.12}$$
This inequality is a consequence of viewing the solution $u(t)$ of Problem (DNLE-d) with initial data u_0 as a semigroup $u(t) = T_t(u_0)$. As an immediate consequence we observe that
$$\|u_t(x,t)\|_1 \leq \mu t^{-1} \|u_0\|_1 \tag{1.13}$$
In addition, using estimate (1.12) with $t/2$ instead of t and s one can obtain that

$$\|u_t(x,t)\|_1 \leq \mu t^{-1} \|u(t/2)\|_1. \tag{1.14}$$

2. Smoothing effects

In the case $m(p-1) > 1$, the solution u of Problem (DNLE-d) satisfies

$$\|u(t,\cdot)\|_{L^r(\Omega)} \le Ct^{-\mu},\ t \in (0,+\infty),\ r \ge 1. \tag{1.15}$$

From the previous estimates we obtain the absolute bound

$$\|u(x,t)\|_{L^\infty(\Omega)} \le Ct^{-\mu}, t \in (0,+\infty). \tag{1.16}$$

As a consequence, we can improve the estimate (1.12) by using (1.16) and we get

$$\|u_t(x,t)\|_{L^1(\Omega)} \le C\mu t^{-1}(t/2)^{-\mu} \le Ct^{-1-\mu}. \tag{1.17}$$

1.2.2 Asymptotic behaviour: Uniform convergence of the rescaled solution to the asymptotic profile

Now we are ready to state the first important result of the paper.

Theorem 1.2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$. Let $u(t,\cdot)$ be a weak solution to Problem DNLE-d corresponding to the nonnegative initial datum $u_0 \in L^1(\Omega)$. Then there exists a unique self-similar solution of Problem* (DNLE-d)

$$U(x,t) = t^{-\mu}f(x),\ t \in (0,+\infty), x \in \Omega.$$

Moreover

$$\lim_{t\to+\infty} t^\mu |u(x,t) - U(x,t)| = \lim_{t\to+\infty} |t^\mu u(x,t) - f(x)| = 0, \tag{1.18}$$

unless u is trivial, $u \equiv 0$. The convergence is uniform in space and monotone non-decreasing in time. Moreover, the asymptotic profile f is the unique non-negative solution of the stationary problem:

$$\Delta_p f^m(x) + \mu f(x) = 0,\ x \in \Omega,\ \ f(x) = 0,\ x \in \partial\Omega. \tag{1.19}$$

Proof. 1**. The main tools** are the a-priori estimates (1.16) and (1.10). These estimates make sense also in the classical way since the weak solution u of Problem (DNLE-d) is in fact locally Hölder continuous.

2**. Rescaled orbit and equation.** We use the rescaling

$$u(x,t) = t^{-\mu}v(x,\tau),\ t = e^\tau. \tag{1.20}$$

As we have seen, Problem (DNLE-d) is mapped into the *rescaled problem*:

$$\begin{cases} v_\tau(x,\tau) = \Delta_p v^m(x,\tau) + \mu v(x,\tau) & \text{for } \tau \in \mathbb{R} \text{ and } x \in \Omega, \\ v(x,0) = v_0(x) = u(x,1) & \text{for } x \in \Omega, \\ v(x,\tau) = 0 & \text{for } \tau \in \mathbb{R} \text{ and } x \in \partial\Omega. \end{cases} \tag{1.21}$$

For this problem, we take zero Dirichlet boundary data in the sense that $v(x,\tau) \in W_0^{1,p}(\Omega)$. The initial data are taken non-negative and integrable in Ω. The possibility of delaying the time origin and the regularity theory allow us to assume that $v(x,0)$ is bounded, even continuous.

3. **Convergence.** The advantage of the new variable is seen when we translate the estimates information in terms of v. From the a-priori estimates (1.16) and (1.14) we get better estimates for the function v:

$$0 \le v \le C, \tag{1.22}$$

and

$$v_\tau \ge 0. \tag{1.23}$$

We conclude from this that for every $x \in \Omega$ there exists the limit

$$\lim_{\tau\to\infty} v(x,\tau) = f(x)$$

and this convergence is monotone non-decreasing. Also, from (1.22), the function $f(x)$ is nontrivial and bounded. Moreover, by (Beppo Levi's) Monotone Convergence Theorem we have

$$v(\tau,\cdot) \to f \text{ strongly in } L^1(\Omega).$$

Since there is an $L^\infty(\Omega)-$bound the convergence takes place in all $L^q(\Omega)$, $1 \le q \le p$. This function f is called an **asymptotic profile** and we will prove that it is the solution of the stationary elliptic problem associated to the rescaled problem (1.21) and that it is unique.

4. **Energy estimates.** We consider the next Lyapunov functional, called **entropy**,

$$E(\tau) = E[v(\tau)] := \frac{1}{p}\int_\Omega |\nabla v^m(x,\tau)|^p dx - \frac{m}{m+1}\mu\int_\Omega v^{m+1}(x,\tau)dx.$$

We compute the entropy dissipation (Fisher information)

$$\frac{d}{d\tau}E(\tau) = -m\int_\Omega v^{m-1}(x,\tau)v_\tau^2(x,\tau)dx \le 0,$$

which means that $E(\tau)$ is a non-increasing function. Then we obtain that

$$E(\tau) \le E(0) = \frac{1}{p}\int_\Omega |\nabla v_0^m(x)|^p dx - \frac{m}{m+1}\mu\int_\Omega v_0^{m+1}(x)dx. \tag{1.24}$$

From this energy estimate (1.24) together with the a priori estimate (1.22) we get that $|\nabla v^m(t,\cdot)|$ is uniformly bounded in time in every $L^q(\Omega)$-norm, $1 \le q \le p$, thus weakly convergent up to subsequences. Let us denote by M the uniform bound for the $L^p(\Omega)$-norm:

$$\int_\Omega |\nabla v^m(x,\tau)|^p dx \le M, \quad \forall t \in \mathbb{R}. \tag{1.25}$$

We deduce that

$$\partial_{x_i} v(\tau, \cdot) \rightharpoonup \partial_{x_i} f(\cdot) \quad \text{when } \tau \to \infty \text{ in } L^q(\Omega), \qquad \text{for every } 1 < q \leq p. \tag{1.26}$$

We can also get a uniform bound for $\|v_\tau\|_{L^1(\Omega)}$, using the Bénilan-Crandall type estimate (1.17) as follows:

$$\|v_\tau(x,\tau) - \mu v(x,\tau)\|_{L^1(\Omega)} = \|\Delta_p v^m(x,\tau)\|_{L^1(\Omega)} = \|\Delta_p \left(t^\mu u(x,t)\right)^m\|_{L^1(\Omega)}$$

$$= t^{1+\mu}\|\Delta_p u^m(x,t)\|_{L^1(\Omega)} = t^{1+\mu}\|u_t(x,t)\|_{L^1(\Omega)} \leq C_1.$$

Thus, by now we have:

$$\|v_\tau - \mu v\|_{L^1(\Omega)} \leq C_1$$

and then we obtain a uniform bound in time

$$\|v_\tau\|_{L^1(\Omega)} \leq C_2. \tag{1.27}$$

Remark: in the PLE case these estimates can be obtained also in $L^p(\Omega)$ because of the contractivity property of the p-Laplacian operator in $L^p(\Omega)$.

5. **Convergence in measure of gradients.** The weak convergence of ∇v^m to ∇f^m can be improved, and in fact we will prove a stronger result, the convergence in measure (with respect to the Lebesgue measure $\mathcal{L}$). The idea comes from [16]. We refer also to [27] where the authors prove this result for the fast p-Laplacian equation. A strong argument in our proof are the following inequalities for vectors in $\mathbb{R}^n$. If $2 \leq p$ then

$$< |a|^{p-2}a - |b|^{p-2}b, a - b > \geq \gamma_1 |a-b|^p, \quad \forall a, b \in \mathbb{R}^N, \tag{1.28}$$

where $\gamma_1 = c_p$ is a constant depending on p and n. If $1 < p < 2$ then

$$< |a|^{p-2}a - |b|^{p-2}b, a - b > \geq \gamma_2 \frac{|a-b|^2}{|a|^{2-p} + |b|^{2-p}}, \tag{1.29}$$

with optimal constant $\gamma_2 = c_p = \min\{1, 2(p-1)\}$. For a proof of these inequalities we refer to [27, 55].

We prove now that $\nabla v^m(\tau, \cdot)$ converges in measure to $\nabla f^m(\cdot)$ when $\tau \to \infty$. We remark that it is sufficient to prove that $(\nabla v^m(\tau, \cdot))_{\tau > 0}$ is Cauchy in measure. Thus we have to prove that for every $\epsilon > 0$ and $A > 0$ there exists $T > 0$ and such that

$$\mathcal{L}\left(\{x \in \Omega : |\nabla v^m(x,\tau_1) - \nabla v^m(x,\tau_2)| > A\}\right) < \epsilon, \ \forall \tau_1, \tau_2 > T.$$

Let $\epsilon > 0$, $A > 0$ and S be the set whose measure we want to estimate

$$S := \{x \in \Omega : |\nabla v^m(x,\tau_1) - \nabla v^m(x,\tau_2)| > A\}.$$

Then $S \subset S_1 \cup S_2$, where

$$\begin{aligned} S_1 =& \{x \in \Omega : |\nabla v^m(x,\tau_1)| > A\} \cup \{x \in \Omega : |\nabla v^m(x,\tau_2)| > A\}, \\ S2 =& \{x \in \Omega : |\nabla v^m(x,\tau_1)| \le A,\ |\nabla v^m(x,\tau_1)| \le A,\ |\nabla v^m(x,\tau_1) - \nabla v^m(x,\tau_2)| > A\}. \end{aligned}$$

Since $|\nabla v^m(\tau, x)|$ is uniformly bounded in $L^p(\Omega)$, then $\mathcal{L}(S_1) < \epsilon$ for τ_1, τ_2 sufficiently large. Now, in order to estimate $\mathcal{L}(S_2)$, the idea is to use the algebraic inequalities (1.28) and (1.29) for the vectors $\nabla v^m(\tau_1)$ and $\nabla v^m(\tau_2)$.

• If $p \ge 2$ then

$$\begin{aligned} &S_2 \subset \{x \in \Omega : |\nabla v^m(x,\tau_1)| \le A,\ |\nabla v^m(x,\tau_1)| \le A, \\ &(|\nabla v^m(\tau_1)|^{p-2}\nabla v^m(\tau_1) - |\nabla v^m(\tau_2)|^{p-2}\nabla v^m(\tau_1)) \cdot (\nabla v^m(\tau_1) - \nabla v^m(\tau_2)) \ge \gamma_1 \lambda^p := \beta_1\}. \end{aligned}$$

• If $1 < p < 2$ then

$$\begin{aligned} S_2 \subset& \{x \in \Omega : |\nabla v^m(x,\tau_1)| \le A,\ |\nabla v^m(x,\tau_1)| \le A, \\ & (|\nabla v^m(t_1)|^{p-2}\nabla v^m(\tau_1) - |\nabla v^m(\tau_2)|^{p-2}\nabla v^m(\tau_1)) \cdot (\nabla v^m(\tau_1) - \nabla v^m(\tau_2)) \\ & \ge \gamma_2 \frac{\lambda^2}{2A^{2-p}} =: \beta_2\}. \end{aligned}$$

We remark that in the particular case of the degenerate PLE we only have to consider the case $p > 2$.

Now, for $\beta = \beta_1$ if $p \ge 2$, respectively $\beta = \beta_2$ if $1 < p < 2$, we obtain that

$$\begin{aligned} \mathcal{L}(S_2) &= \int_{S_2} d\mu \\ &\le \frac{1}{\beta}\int_\Omega (|\nabla v^m(\tau_1)|^{p-2}\nabla v^m(\tau_1) - |\nabla v^m(\tau_2)|^{p-2}\nabla v^m(t_1)) \cdot (\nabla v^m(\tau_1) - \nabla v^m(\tau_2))dx \\ &= -\frac{1}{\beta}\int_\Omega \left[\nabla.(|\nabla v^m(\tau_1)|^{p-2}\nabla v^m(\tau_1) - |\nabla v^m(\tau_2)|^{p-2}\nabla v^m(\tau_1))\right] \left[v^m(\tau_1) - v^m(\tau_2)\right] dx \\ &= -\frac{1}{\beta}\int_\Omega (\Delta_p v^m(\tau_1) - \Delta_p v^m(\tau_2))(v^m(\tau_1) - v^m(\tau_2))dx \\ &= -\frac{1}{\beta}\int_\Omega \left(v_\tau(\tau_1) + \mu v(\tau_1) - v_\tau(\tau_2) - \mu v(\tau_2)\right)(v^m(\tau_1) - v^m(\tau_2))dx, \end{aligned}$$

where we used integration by parts. We recall that v is positive and uniformly bounded in time by (1.22) and the norm $\|v_\tau\|_{L^2(\Omega)}$ is also uniformly bounded in time by (1.27). Thus

$$\mathcal{L}(S_2) < \frac{1}{\beta}C$$

where C is a constant positive number and it follows that

$$\mathcal{L}(S_2) < \epsilon,$$

for β big enough.

Thus, we proved that the sequence $(\nabla v^m(\tau,\cdot))_{\tau>0}$ is Cauchy in measure, thus it converges in measure to a function $W:\Omega\to\mathbb{R}^N$. It is well a known fact (Lemma 1.6.1 in the Appendix) that if a sequence is uniformly bounded in $L^p(\Omega)$ and converges in measure, it converges strongly in any $L^q(\Omega)$, for any $1\le q<p$. It follows that

$$\nabla v^m(x,\tau)\to W(x) \quad \text{in } (L^q(\Omega))^N \text{ when } \tau\to\infty, \text{ for every } 1\le q<p.$$

Since we already know the weak convergence (1.26) we get that $w=\nabla f^m$ a.e. in Ω and we can conclude that

$$\nabla v^m(x,\tau)\to\nabla f^m(x) \quad \text{in measure}$$

and

$$\nabla v^m(x,\tau)\to\nabla f^m(x) \quad \text{in } (L^q(\Omega))^N, \text{ for every } 1\le q<p. \tag{1.30}$$

Thus, we get that, up to subsequences,

$$\nabla v^m(x,\tau)\to\nabla f^m(x) \quad \text{a.e. in } \Omega. \tag{1.31}$$

6. **The limit is a stationary solution.** Multiply equation (1.21) by any test function $\phi(x)\in C_c^\infty(\Omega)$ and integrate in space, $x\in\Omega$, and time between τ_1 and $\tau_2=\tau_1+T_0$, for a fixed $T_0>0$. We get that

$$\begin{aligned}\int_\Omega v(\tau_2)\phi dx-\int_\Omega v(\tau_1)\phi dx &= \int_{\tau_1}^{\tau_2}\int_\Omega \Delta_p v^m\phi dxdt+\mu\int_{\tau_1}^{\tau_2}\int_\Omega v\phi dxdt=\\ &=-\int_{\tau_1}^{\tau_2}\int_\Omega|\nabla v^m|^{p-2}\nabla v^m\cdot\nabla\phi dxdt+\mu\int_{\tau_1}^{\tau_2}\int_\Omega v\phi dxdt.\end{aligned}$$

Let $\tau_1\to\infty$. Then also $\tau_2\to\infty$ and we get that

$$\int_\Omega v(\tau_2)\phi dx-\int_\Omega v(\tau_1)\phi dx\to 0$$

since $v(\tau)\to f$ in $L^1(\Omega)$ when $\tau\to\infty$ and $\phi\in C_c^\infty(\Omega)$. Then, by Lebesgue's Dominated Convergence Theorem, it follows that

$$\mu\int_{\tau_1}^{\tau_2}\int_\Omega v\phi dxdt\to T_0\mu\int_\Omega f\phi dx.$$

Now, we need to obtain the convergence of the integrals involving gradients:

$$\int_{\tau_1}^{\tau_2}\int_\Omega|\nabla v^m|^{p-2}\nabla v^m\cdot\nabla\phi dxdt\to T_0\int_\Omega|\nabla f^m|^{p-2}\nabla f^m\cdot\nabla\phi dx \tag{1.32}$$

when $\tau_1\to\infty$ in order to obtain that

$$0=-T_0\int_\Omega|\nabla f^m|^{p-2}\nabla f^m\cdot\nabla\phi dx+\mu T_0\int_\Omega f\phi dx,$$

and dividing by T_0 and integrating by parts we get

$$0 = \int_\Omega \Delta_p f^m \phi dx + \mu \int_\Omega f \phi dx,$$

which proves that f is a weak solution of the stationary problem (1.19).

In order to justify assertion (1.32), we remark that, after a change of variables, the left term can be written as

$$\int_0^{T_0} \int_\Omega |\nabla v^m(x, \tau + \tau_1)|^{p-2} \nabla v^m(x, \tau + \tau_1) \cdot \nabla \phi dx dt.$$

Thus, it is enough to prove that

$$\int_0^{T_0} \int_\Omega |\nabla v^m(x, \tau + n)|^{p-2} \nabla v^m(x, \tau + n) \cdot \nabla \phi \, dx dt \to T_0 \int_\Omega |\nabla f^m|^{p-2} \nabla f^m \cdot \nabla \phi \, dx$$

when $n \to \infty$. The idea will be to use Lemma 1.6.2 from the Appendix in the following context. Let $\Omega_1 = \Omega \times [0, T_0)$ (finite measure space), $H = \mathbb{R}^N$ (Hilbert space) and let

$$A : H \to H, \quad A(Z) = |Z|^{p-2} Z.$$

Then A is single valued, monotone and $R(A + I) = H$ and therefore, according to the theory of monotone operators, A is maximal monotone. We consider the sequences

$$Z_n(x, t) = \nabla v^m(x, \tau + n) : \Omega_1 \to H,$$

$$W_n(x, \tau) = A(Z_n(x, \tau)) = |\nabla v^m(x, \tau + n)|^{p-2} \nabla v^m(x, \tau + n) : \Omega_1 \to H.$$

The hypothesis of Lemma 1.6.2 are satisfied as follows. From (1.31)

$$Z_n(x, \tau) \to Z(x, \tau) = \nabla f^m(x) \text{ a.e. on } \Omega_1.$$

Let $q = \frac{p}{p-1}$. Then $W_n(x, t)$ is uniformly bounded in $L^q(\Omega_1; H)$, by the energy estimate (1.25), since

$$\int_0^{T_0} \int_\Omega ||\nabla v^m(x, \tau + n)|^{p-2} \nabla v^m(x, \tau + n)|^q dx dt \le M T_0, \ \forall n > 0,$$

and thus it converges weakly (up to subsequences) to a function W in $L^q(\Omega_1; H)$ when $n \to \infty$. But since Ω_1 is bounded, then weak convergence in $L^q(\Omega_1; H)$ implies the weak convergence in $L^1(\Omega_1; H)$ and thus

$$W_n(x, \tau) \rightharpoonup W(x, \tau) \text{ weakly in } L^1(\Omega_1; H).$$

Now, according to Lemma (1.6.2), we obtain that

$$W(x,\tau) = A(Z(x,\tau)) = |\nabla f^m(x)|^{p-2}\nabla f^m(x).$$

Thus, the weak limit of $W_n(x,\tau)$ is unique and we get that

$$|\nabla v^m(x,\tau+n)|^{p-2}\nabla v^m(x,\tau+n) \rightharpoonup |\nabla f^m(x)|^{p-2}\nabla f^m(x) \text{ weakly in } L^q(\Omega_1;H).$$

By taking ϕ smooth and compactly supported in Ω we obtain the desired convergence

$$\begin{aligned}\int_0^{T_0}\int_\Omega |\nabla v^m(x,\tau+n)|^{p-2}\nabla v^m(x,\tau+n)\nabla\phi dxdt \to \int_0^{T_0}\int_\Omega |\nabla f^m|^{p-2}\nabla f^m\nabla\phi dxdt\\ = T_0\int_\Omega |\nabla f^m|^{p-2}\nabla f^m\nabla\phi dx.\end{aligned}$$

7. **Uniqueness of the stationary solution.** Let us prove that the nonnegative and nontrivial stationary solution is unique. If we have two stationary solutions of (1.19), f_1 and f_2, we can construct solutions of the (DNLE) of the form

$$U_1(x,t) = t^{-\mu}f_1(x), \quad U_2(x,t) = (t+s)^{-\mu}f_2(x),$$

for some $s>0$. U_2 has initial data $U_2(x,0) = s^{-\mu}f_2(x)$. Formally, $U_1(x,0)$ has infinite values and then by the Comparison Principle we conclude that $U_2(x,t) \le U_1(x,t)$. The technical details of the proof are as follows: by the L^1-dependence theorem of weak solutions of Problem (DNLE-d) we know that

$$\int_\Omega (U_2(x,t) - U_1(x,t))_+ dx,$$

is decreasing in time. The proof of this fact is standard: we perform the difference of the equations of U_1 and U_2, multiplying by $h(w)$, where $w = U_1^m - U_2^m$ and h is a $C^1(\mathbb{R})$ function such that $0 \le h \le 1$, $h(s) = 0$ for $s \le 0$ and $h'(s) > 0$ for $s \ge 0$, and then integrate on Ω. Notice that

$$0 \le \mu_0 := \inf_{x\in\Omega} h'(w(x)) < \infty.$$

Because of the nonlinearity of the $p-$laplacian operator, we will use again the algebraic inequalities (1.28) and (1.29).

$$\begin{aligned}\int_\Omega (U_2(x,t) - U_1(x,t))_t\, h(w)dx &= \int_\Omega (\Delta_p U_2^m(x,t) - \Delta_p U_1^m(x,t))h(w)dx\\
&= -\int_\Omega \left(|\nabla U_1^m|^{p-2}\nabla U_1^m - |\nabla U_2^m|^{p-2}\nabla U_2^m\right)\cdot\nabla h(w)dx\\
&= -\int_\Omega \left(|\nabla U_1^m|^{p-2}\nabla U_1^m - |\nabla U_2^m|^{p-2}\nabla U_2^m\right) h'(w)\cdot\nabla(U_1^m - U_2^m)dx\\
&\le -\mu_0\int_\Omega \left(|\nabla U_1^m|^{p-2}\nabla U_1^m - |\nabla U_2^m|^{p-2}\nabla U_2^m\right)\cdot\nabla(U_1^m - U_2^m)dx.\end{aligned}$$

Now, if $p \geq 2$ we obtain that

$$\int_\Omega (U_2(x,t) - U_1(x,t))_t \ h(w)dx \leq -\mu_0\gamma_1 \int_\Omega |\nabla U_1^m - \nabla U_2^m|^p dx \leq 0,$$

and if $1 < p < 2$ the estimate will be

$$\int_\Omega (U_2(x,t) - U_1(x,t))_t \ h(w)dx \leq -\mu_0\gamma_2 \int_\Omega \frac{|\nabla U_1^m - \nabla U_2^m|^2}{|\nabla U_1^m|^{2-p} + |\nabla U_2^m|^{2-p}} dx \leq 0.$$

Letting h converge to the sign function sign_0^+ we get that

$$\frac{d}{dt}\int_\Omega (U_2(x,t) - U_1(x,t))_+ dx \leq 0.$$

Now, the integral goes to 0 as $t \to 0$ because $U_1(x,t)$ goes pointwise to infinity as $t \to 0$ and then $U_2(x,t) > U_1(x,t)$ for t large enough. We conclude that $(U_2(x,t) - U_1(x,t))_+ = 0$ a.e. in x for every $t > 0$. Thus $U_2(x,t) \leq U_1(x,t)$ a.e. in x for every $t > 0$. Using the form of U_1 and U_2, we get

$$f_2(x) \leq \left(\frac{t+s}{t}\right)^\mu f_1(x).$$

Letting $s \to 0$ we get $f_2(x) \leq f_1(x)$. The converse inequality is similar.

8. **Better convergence.** We have established the result (1.18) in the sense of $L^1(\Omega)$ convergence. The passage to uniform convergence depends on having better regularity for the solutions , i.e. on a compactness argument. It is known that uniformly bounded solutions of the (DNLE) are C^α continuous in space and time with uniform Hölder exponent and coefficients (see (1.130) in the Appendix A).

Consider now the second type of rescaling that we may call *fixed-rate rescaling*

$$u_\lambda(x,t) = \lambda^\mu u(x, \lambda t). \tag{1.33}$$

For every $\lambda > 0$ the function u_λ is still a solution of the (DNLE) to which the a-priori estimate (1.16) applies. Hence, in a set of the form $\Omega \times (1,2)$ this family is equi-continuous and by Ascoli-Arzela Theorem it converges along a subsequence $\lambda_j \to \infty$. Now, observe that

$$u_\lambda(x,1) = v(x, \log\lambda)$$

to conclude that $v(x, \log\lambda_j)$ converges uniformly. Since the limit is fixed, f, the whole family $v(x,\tau)$ converges as $\tau \to \infty$ and (1.18) is proved. □

1.2.3 Some remarks on the asymptotic profile

1. **Existence.** The proof of Theorem 1.2.1 also guarantees the existence of a solution of the stationary problem (1.19) by obtaining it as the limit of $v(t,\cdot)$ when t goes to ∞. As we have previously established, this solution is called the asymptotic profile of parabolic problem (DNLE-d).

Furthermore, we recall a second proof of existence, based on an *entropy method*, which can be applied also for the general case of solutions with changing sign.

The elliptic problem (1.19) can be written in terms of the function $w = f^m$ (the notation makes sense since $f > 0$ in Ω by Maximum Principle) as

$$\Delta_p w(x) + \mu w^{\frac{1}{m}}(x) = 0,\ x \in \Omega,\ \ w(x) = 0,\ x \in \partial\Omega. \tag{1.34}$$

The typical approach to solving equation (1.34) for the experts in elliptic equations is to view the solution w as a critical point of the functional

$$J(g) = \frac{1}{p}\int_\Omega |\nabla w|^p dx - \frac{m}{m+1}\mu \int_\Omega w^{\frac{m+1}{m}} dx. \tag{1.35}$$

The proof is classical and we resume it following the ideas from [104]. It can be showed that J is well defined in $W_0^{1,p}(\Omega)$ since $m(p-1) > 1$, J is bounded from below via Poincare's Inequality, and also the infimum is negative. Moreover, along any minimizing sequence there is convergence in $W_0^{1,p}(\Omega)$ and the infimum is taken, hence J has a minimum. Also,

$$J(g) \geq J(w), \quad \forall g \in W_0^{1,p}(\Omega),$$

where w is the solution of (1.34) and it follows that w is the point where J attains its minimum.

2. **Uniqueness.** We already proved the uniqueness of the asymptotic profile in point 7 of the proof of Theorem 1.2.1.

3. **Regularity.** We know that w is a bounded solution of the equation

$$\Delta_p w(x) + \mu w^{\frac{1}{m}}(x) = 0.$$

By known regularity results for degenerate elliptic equations, we get that $w \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1]$.

4. **Behaviour near the boundary.** Concerning the behaviour of f, the following estimates were proved in [86]:

$$|\nabla f| \leq C_0 d(x)^{1/m-1}, \quad \forall x \in \Omega, \tag{1.36}$$

and

$$C_1 d(x)^{1/m} \leq f(x) \leq C_2 d(x)^{1/m}, \quad \forall x \in \Omega. \tag{1.37}$$

that is $w(x) = f^m(x)$ has a linear growth near the boundary.

Also, w satisfies a Boundary Principle

$$\frac{\partial w}{\partial \nu}(x) \equiv \nabla w(x) \cdot \nu(x) < 0, \quad \text{for } x \in \partial\Omega, \tag{1.38}$$

where $\nu(x)$ denotes the outward unit normal vector to $\partial\Omega$ at the point x. Notice that, in the case of the PLE, the boundary principle is satisfied by f.

1.3 Rate of Convergence for $m(p-1) > 1$. Proof of Theorem 1.1.1

In this section we will give the proof of Theorem 1.1.1. We previously showed in Theorem 1.2.1 that for any initial datum $u_0 \in L^1(\Omega)$, the corresponding rescaled solution $t^\mu u(x,t)$ converges to an unique asymptotic profile f uniformly in space and monotonically nondecreasing in time. The goal of this section is to provide sharp convergence rates, namely to prove that

$$|(1+t)^\mu u(x,t) - f(x)| \le \mathcal{C} f(x)(1+t)^{-1} \quad \text{for all } t \ge t_0,\ x \in \overline{\Omega}, \tag{1.39}$$

where f is the solution of the elliptic problem (1.19).

The proof of this result uses ideas introduced by Aronson and Peletier for the (PME) in [8]. Although a similar proof can be adapted to the case of the (DNLE) with some lengthly arguments, in this work we will give a simpler proof based on the results of Section 1.2. Let us explain the strategy of the proof.

1. Improved upper bound. In Theorem 1.3.1 we prove that there exists a constant $s_1 > 0$ depending only on p, m, N, u_0 and Ω such that

$$0 \le u(x,t) \le (s_1 + t)^{-\mu} f(x), \quad \forall x \in \Omega,\ t \ge 1.$$

2. Positivity. In Proposition 1.3.2 we prove that even if u_0 has compact support there exists $T' > 0$ depending only on p, m, N, u_0 and Ω such that

$$u(x,t) > 0, \quad \forall x \in \Omega,\ t > T'.$$

3. Sharp lower bound. In Theorem 1.3.2 we prove that there exist $T'' \ge 0$ and $s_0 > 0$ depending only on p, m, N, u_0 and Ω such that

$$u(x,t) \ge (s_0 + t)^{-\mu} f(x), \quad \forall x \in \overline{\Omega},\ t \ge T''.$$

Then, the estimate (1.39) follows as a consequence of the upper and lower bounds together with the boundedness of the asymptotic profile $0 \le f \le C$.

1.3.1 Reduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$. Let $u(x,t)$ be a weak solution to the Problem (DNLE-d) corresponding to the nonnegative initial datum $u_0 \in L^1(\Omega)$. Previously, in Theorem 1.2.1, we proved that the rescaled solution $t^\mu u(x,t)$ is monotone increasing in time and convergent to the function $f(x)$, thus bounded from above by $f(x)$:

$$u(x,t) \le t^{-\mu} f(x), \quad \forall t \in (0,\infty),\ x \in \overline{\Omega}. \tag{1.40}$$

In virtue of this result we can assume that the data satisfy the following conditions, denoted as **Hypothesis (H)**:

(H1) $\Omega \subseteq \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$.

(H2) u_0 is a nonnegative function defined on $\overline{\Omega}$ such that $u_0 \in L^1(\Omega)$, $u_0 = 0$ on $\partial\Omega$ and there exists $s_1 > 0$ such that

$$u_0(x) \le s_1 f(x), \quad \forall x \in \overline{\Omega}. \tag{1.41}$$

Since the DNLE is invariant under time displacement then (H2) is satisfied by starting with initial data $u(t_0,\cdot)$, where $t_0 > 0$. We assume henceforth that such a displacement in time has been done.

1.3.2 Improved upper bound for u

In the following theorem, we will improve the upper bound (1.40) of u that we have previously proved in Theorem 1.2.1.

Theorem 1.3.1. ***(Quantitative upper bound)*** *Assume that Ω and u_0 satisfy the hypothesis (H), and let u be the corresponding solution of the Problem* (DNLE-d). *Then there exists a constant $s_1 > 0$ such that*

$$u(x,t) \le (s_1 + t)^{-\mu} f(x), \quad \forall t \ge 0,\ x \in \overline{\Omega}. \tag{1.42}$$

where s_1 depends on p, m, N, u_0 and Ω.

Proof. It relies on the Comparison Principle for the (DNLE). Consider as comparison function the separate variable solution U of the (DNLE) given by

$$U(x,t) := U(t,x;s_1) = (s_1 + t)^{-\mu} f(x),$$

where $s_1 > 0$ is the constant given in the Hypothesis (H2) which satisfies the inequality (1.41)

$$u_0(x) \le U(x,0) = s_1^{-\mu} f(x), \quad \forall x \in \overline{\Omega}.$$

Therefore by comparison, it follows that

$$u(x,t) \le U(x,t), \quad \forall t \ge 0,\ x \in \overline{\Omega}.$$

$\square$

1.3.3 Positivity of u

In this subsection we prove the positivity of u in Ω, under the Hypothesis (H). This will be done in two steps. First, in Proposition 1.3.1, we will prove the positivity of u in a domain $\Omega_{I,\delta} \subset \Omega$ and then we complete the result by proving positivity of u up to the boundary in Proposition 1.3.2. In this direction, we will make use of the properties of the distance to the boundary function $d(x) = d(x, \partial\Omega)$ stated in Subsection 1.6.3 of the Appendix A. In terms of $d(x)$ we define the following sets

$$\Omega_{I,r} = \{x \in \overline{\Omega} : d(x) > r\}, \quad \Omega_r = \Omega \setminus \overline{\Omega_{I,r}} = \{x \in \overline{\Omega} : d(x) < r\}.$$

Throughout the paper we will use the notation ξ_0 for the critical value which implies good properties of the distance to the boundary function in Ω_{ξ_0} as we explain in Subsection 1.6.3 of the Appendix.

Proposition 1.3.1. ***(Inner positivity)*** *Assume that Ω and u_0 satisfy (H) and let u denote the weak solution of Problem* (DNLE-d) *and f denote the solution of Problem* (1.19)*. Let $0 < 2\delta < \xi_0$ fixed, where ξ_0 is defined in Lemma 1.6.3. Then there exist $\epsilon > 0$ and $T_1 \in [0, +\infty)$ such that*

$$u(x, T_1) > \epsilon \textit{ for all } x \in \Omega_{I,\delta},$$

where ϵ and T_1 depend only on m, p, N, Ω and u_0.

Proof. The main tools are the uniform convergence (Theorem 1.2.1) of the rescaled solution

$$v(\tau, x) = t^{\mu} u(x, t), \quad t = e^{\tau},$$

defined in (1.20) to the asymptotic profile f and the properties of f given by (1.37). More exactly, there exists $C_1, C_2 > 0$ such that

$$C_1 d^{\frac{1}{m}}(x) \le f(x) \le C_2 d^{\frac{1}{m}}(x), \ \forall x \in \Omega. \tag{1.43}$$

Let $\epsilon_0 = \frac{C_1}{2}\delta^{\frac{1}{m}} > 0$. Then there exists $T_0 \ge 0$ such that

$$\|v(x,\tau) - f(x)\|_{L^\infty(\Omega)} \le \epsilon_0, \quad \forall \tau \ge T_0.$$

Since the convergence of $v(x,\tau)$ to $f(x)$ is monotone nondecreasing in τ we derive that

$$v(x,\tau) \ge f(x) - \epsilon_0, \quad \forall x \in \Omega, \ \forall \tau \ge T_0,$$

and then, by using (1.43), for $x \in \Omega_{I,\delta}$ we obtain the lower bound

$$v(x,\tau) \ge C_1 d^{\frac{1}{m}}(x) - \epsilon_0 \ge C_1 \delta^{\frac{1}{m}} - \epsilon_0 = \epsilon_0.$$

In terms of $u(x,t)$ these estimates rewrites as

$$u(x,t) \geq \epsilon_0 t^{-\mu}, \quad \forall x \in \Omega_{I,\delta},\ \forall t \geq e^{T_0}.$$

Let

$$T_1 = e^{T_0}, \quad \epsilon = \epsilon_0 T_1^{-\mu} = \epsilon_0 e^{-T_0 \mu}. \tag{1.44}$$

Then

$$u(x, T_1) \geq \epsilon, \quad \forall x \in \Omega_{I,\delta}.$$

□

Proposition 1.3.2. ***(Positivity up to the boundary)*** *Assume that Ω and u_0 satisfy the hypothesis (H) and let u denote the weak solution of the problem* (DNLE-d) *and f denote the solution of the problem* (1.19). *Consider $T_1 > 0$ given by Proposition 1.3.1. Then there exists $T_2 > 0$*

$$u(x, T_1 + T_2) > 0, \quad \forall x \in \Omega,$$

and T_2 depends only on m, p, N, Ω and u_0.

Proof. We consider $0 < 2\delta < \xi_0$ as in Proposition 1.3.1. First, by (1.133), we observe that

$$\Omega_{2\delta} \subset \bigcup_{\{y \in \partial\Omega_{I,2\delta}\}} B_{2\delta}(y).$$

Then, since we have already proved the positivity inside the domain in Proposition 1.3.1, it is enough to demonstrate that there exists $T_2 \geq 0$ such that

$$u(x, T_1 + T_2) > 0, \quad \forall x \in B_{2\delta}(y),\ \forall y \in \partial\Omega_{I,2\delta}.$$

Let ϵ given by (1.44). Let $y \in \partial\Omega_{I,2\delta}$ and consider the Barenblatt solution $\mathcal{U}$ as in Section 1.4 such that

$$\operatorname{supp} \mathcal{U}(x-y, 0; a, s) = \overline{B}_\delta(y) \ \text{ and } \ \max \mathcal{U}(x-y, 0; a, s) = \epsilon,$$

that is

$$a = \left(\frac{\epsilon}{c}\delta^N\right)^{\beta(m(p-1)-1)}, \quad s = \delta^p \left(\frac{c}{\epsilon}\right)^{m(p-1)-1}. \tag{1.45}$$

Now, assume that $T_2 = t$ is the time when $\operatorname{supp} \mathcal{U}(x-y, t; a, s)$ reaches the boundary of Ω, that is when

$$\operatorname{supp} \mathcal{U}(x-y, t; a, s) = \overline{B}_{2\delta}(y).$$

This implies

$$T_2 = \left(\frac{2\delta}{a}\right)^{1/\beta} - s. \tag{1.46}$$

We want to apply now the Parabolic Comparison Principle. To this aim we need to compare $u(x, T_1 + t)$ and $\mathcal{U}(x - y, t; a, s)$ when (x, t) belongs to the parabolic boundary $B_{2\delta} \times \{0\} \cup \partial B_{2\delta} \times [0, T_2]$. Firstly, for $t = 0$ and $x \in B_{2\delta}$, we have

$$u(x, T_1) \geq \begin{cases} \epsilon \geq \mathcal{U}(x - y, 0; a, \tau), & x \in B_\delta(y); \\ 0 = \mathcal{U}(x - y, 0; a, \tau), & x \in B_{2\delta}(y) \setminus B_\delta(y). \end{cases}$$

Secondly, when $t \in [0, T_2]$ and $x \in \partial B_{2\delta}$ we have

$$u(x, T_1 + t) \geq \begin{cases} \epsilon \geq \mathcal{U}(x - y, 0; a, \tau), & x \in \partial B_{2\delta} \setminus \partial\Omega; \\ 0 = \mathcal{U}(x - y, 0; a, \tau), & x \in \partial B_{2\delta} \cap \partial\Omega. \end{cases}$$

Therefore we obtain that

$$u(x, T_1 + T_2) = \mathcal{U}(x - y, T_1 + T_2; a, s), \quad \forall x \in B_{2\delta}(y).$$

We notice from (1.45) and (1.46) that T_2 does not depend on the point y, but only on the data.

□

1.3.4 Sharp lower bound for u

In the following theorem we will derive a lower bound for u, similar to the upper bound we have previously proved in Theorem 1.3.1.

Theorem 1.3.2. ***(Quantitative lower bound)*** *Assume that Ω and u_0 satisfy the hypothesis (H) and let u be the weak solution of the problem* (DNLE-d) *and f be the solution of the problem* (1.19)*. Then there exist two positive constants $s_0 > 0$ and $T_4 > 0$ such that*

$$u(x, t) \geq (s_0 + t)^{-\mu} f(x), \quad \forall x \in \overline{\Omega}, \ \forall t \in [T_4, +\infty), \tag{1.47}$$

where s_0 and T_4 depend only on m, p, N, Ω and u_0.

Before we start the proof of this theorem, we will establish the following preliminary results.

We will make comparison with so called intermediate subsolutions defined by

$$\mathcal{V}(x - y, t; M, s) = (t + s)^{-\alpha} [g^{\frac{1}{m}} (|x - y|(t + s)^{-\beta})]_+.$$

The functions $\mathcal{V}$ are self-similar solutions of the (DNLE) problem in the whole space, α and β are the self-similarity exponents and $g : [0, \infty) \to \mathbb{R}$ is the corresponding profile function. These functions will be presented in more details in Section 1.4.

Proposition 1.3.3. *Under the assumptions of Theorem 1.3.2, for every $y \in \partial\Omega_{I,2\delta}$, where $0 < 2\delta < \xi_0$ is fixed as in Proposition 1.3.1, we can choose constants M and s*

such that there exists a time $T_3 > 0$, for which the next inequality holds:

$$u(x, T_1 + t) \geq \mathcal{V}(x - y, t; M, s), \quad \forall x \in \overline{\Omega},\ \forall t \in [0, T_3], \tag{1.48}$$

where T_1 is the time we have obtained in Proposition 1.3.1 and $T_3 > 0$ depends only on m, p, N, Ω and u_0, and is independent of y.

Proof. The same ideas as in Proposition 1.3.2 apply, since the functions $\mathcal{V}$ have a behaviour similar to the Barenblatt functions. Consider the selfsimilar subsolution $\mathcal{V}$ such that

$$\operatorname{supp} \mathcal{V}(x - y, 0; M, s) = \overline{B}_\delta(y) \ \text{ and } \ \max \mathcal{V}(x - y, 0; M, s) = \epsilon,$$

that is

$$M = \epsilon s^\alpha, \quad \delta = s^\beta a(M) = s^\beta M^{\frac{m(p-1)-1}{p}} a(1) \quad \text{and} \quad s = \left(\frac{\delta}{a(1)}\right)^p \epsilon^{-(m(p-1)-1)}.$$

The exact values of M and s are not important; what matters is that they depend only on the data and, in particular, they are independent of y.

Now, consider the time $T_3 = t$ when $\operatorname{supp} \mathcal{V}(x - y, t; M, s)$ reaches the boundary of Ω, that is when

$$\operatorname{supp} \mathcal{V}(x - y, t; M, s) = \overline{B}_{2\delta}(y). \tag{1.49}$$

We deduce the explicit value for T_3,

$$T_3 = \left(\frac{2\delta}{a}\right)^{1/\beta} - s. \tag{1.50}$$

Then the Parabolic Comparison Principle can be applied to u and $\mathcal{V}$ on the parabolic domain $\overline{\Omega} \times [T_1, T_1 + T_3]$ as in Proposition 1.3.2 and we obtain that

$$u(T_1 + t) \geq \mathcal{V}(x - y, t; M, s), \quad \forall x \in \Omega,\ \forall t \in [0, T_3].$$

We notice from (1.50) and the above formulas for M and s that T_3 does not depend on the point y, but only on the data. $\square$

We define

$$T_4 = T_1 + T_3 \tag{1.51}$$

where T_1 and T_4 are given by Proposition 1.3.1, respectively Proposition 1.3.3.

Proposition 1.3.4. ***(Boundary behaviour)*** *Under the hypothesis of Theorem 1.3.2, let T_4 as in (1.51) and δ as in Proposition 1.3.1. Then there exists a constant $\omega > 0$ depending only on m, p, N, Ω and u_0 such that*

$$u^m(x, T_4) \geq \omega d(x) \quad \textit{for } x \in \Omega_\delta.$$

Proof. Let $y \in \partial\Omega_{I,2\delta}$. Then by Lemma (1.6.3), there exists a unique $z(y) \in \partial\Omega$ such that $\delta = d(y) = |y - z(y)|$. Let

$$\mathcal{V}(x-y, T_3; M, s) = (T_3+s)^{-\alpha}[g^{\frac{1}{m}}(|x-y|(T_3+s)^{-\beta})]_+$$

be the self-similar subsolution obtained in Proposition 1.3.3 where M and s are given by formulas (1.63). Let $[0, a)$ be the largest interval starting from 0 where $g > 0$. By (1.49) it follows that $a = a(M) = 2\delta(T_3+s)^{-\beta}$. Then, in view of the continuity of g', there exist $k_0 < k_1 < 0$ such that

$$k_0 \leq g'(\eta) \leq k_1, \quad \forall \eta \in [\delta(T_3+s)^{-\beta}, 2\delta(T_3+s)^{-\beta}],$$

and it follows that

$$g(\eta) \geq |k_1|(a-\eta), \quad \forall \eta \in [\delta(T_3+s)^{-\beta}, 2\delta(T_3+s)^{-\beta}]. \tag{1.52}$$

Thus it follows from (1.48) and (1.52) that for every $x \in \Omega$ on the segment between y and $z(y)$ such that $\delta < |x-y| < 2\delta$, that is $d(x) < \delta$, u can be bounded from bellow as follows

$$\begin{aligned}
u^m(x, T_4) &\geq \mathcal{V}^m((x-y, T_3; M, s) = (T_3+s)^{-\alpha m} g(|x-y|(T_3+s)^{-\beta}) \\
&\geq |k_1|(T_3+s)^{-\alpha m}\left(2\delta(T_3+s)^{-\beta} - |x-y|(T_3+s)^{-\beta}\right) \\
&= |k_1|(T_3+s)^{-\alpha m-\beta}(2\delta - |x-y|) \\
&= \omega d(x),
\end{aligned} \tag{1.53}$$

where $\omega = |k_1|(T_3+s)^{-\alpha m-\beta} \in \mathbb{R}^+$ is a constant which depends on the data, but not on y and x. Observe that (1.53) holds for arbitrary $y \in \partial\Omega_{I,2\delta}$ and for all x on the inward directed normal through y, provided that $d(x) \leq \delta$. As we remarked in Lemma 1.6.3, the normal map H_r is a homeomorphism for all $r \in [0, \xi_0)$. Therefore, it follows from (1.53) that

$$u^m(x, T_4) \geq \omega d(x) \quad \text{for } x \in \Omega_\delta. \tag{1.54}$$

□

Proof of Theorem 1.3.2. First we will prove that there exists $k_2 > 0$ such that

$$u(x, T_4) \geq k_2 f(x), \quad \forall x \in \Omega. \tag{1.55}$$

By Proposition 1.3.3 and relation 1.37 u satisfies

$$u^m(x, T_4) \geq \omega d(x) \geq \omega C_2^m f^m(x), \quad \text{for } x \in \Omega_\delta.$$

Moreover, by Proposition 1.3.1 and the boundedness of the profile $0 \le f \le C$ in Ω we obtain that

$$u(x, T_1) \ge \epsilon \ge \frac{\epsilon}{C} f(x), \quad \forall x \in \Omega_{I,\delta}.$$

Then, since $T_1 < T_4$, inequality (1.55) is satisfied with $k_2 = \min\{\epsilon/C, \omega^{1/m} C_2\}$.

Finally, let $U(x,t) = (s_0 + t)^{-\mu} f(x)$ be the separate variables solution of the (DNLE) with initial data $s_0^{-\mu} f$, where s_0 is defined by the relation $s_0^{-\mu} = k_2$. Then

$$u(x, T_4) \ge U(x, 0), \quad \forall x \in \Omega$$

and (1.47) follows by Comparison Principle.
End of proof.

1.4 Study of self-similar solutions for the (DNLE)

In this section we will give a short description of the self-similar solutions of the equation (DNLE) $u_t = \Delta_p u^m$, focusing on the properties useful for the proofs in the paper. A complete analysis of these solutions is beyond the purpose of our paper and it can be the subject of a future work.

For a complete characterization of self similar solutions of the PME we refer to [64, 65]. For the self-similar solutions of the PLE in the case $p > 2$ we refer to [19]. Likewise, for the relation between self-similar solutions of the PLE and those of the PME we make reference to [70].

Self-similar solutions of the (DNLE) equation $U_t = \Delta_p U^m$ are functions of the form

$$\mathcal{U}(x,t) = (t+s)^{-\alpha} h(r), \quad r = |x|(t+s)^{-\beta},$$

where $s \ge 0$ is a constant, α and β are positive parameters related by

$$(m(p-1)-1)\alpha + p\beta = 1. \tag{1.56}$$

The profile $g := h^m : [0, \infty) \to \mathbb{R}$ is a function satisfying the differential equation

$$\alpha h(r) + \beta r h'(r) + \frac{1}{r^{N-1}} \left(r^{N-1} |g'(r)|^{p-2} g'(r)) \right)' = 0, \quad r > 0. \tag{1.57}$$

Self-similar solutions are (possibly signed) solutions of the (DNLE) in the whole space $u_t = \Delta_p u^m$, $x \in \mathbb{R}^N$, $t > 0$. When the support of the positive part of such a function $\mathcal{U}$ is included in Ω then $\mathcal{U}_+$ is a sub-solution of the Dirichlet problem for the (DNLE-d)on bounded domains. For this reason, self-similar solutions are useful to indicate the behaviour of a general solution u of the Dirichlet problem (DNLE-d).

We consider the initial conditions

$$h(0) = M^m, \quad h'(0) = 0. \tag{1.58}$$

The existence of a positive solution of ODE (1.57) with initial conditions (1.58) on an interval $[0, a)$, where $a \in (0, \infty]$, can be proved using fixed point methods when $p \leq 2$ and using phase plane methods when $p > 2$ (we refer to [19] when $p > 2$ where the author discusses the case of the PLE). As far as we know, fixed point methods do not work when $p > 2$.

If one multiplies the ODE by r^{N-1} and integrates between 0 and r, where $r \in [0, a)$, it follows that

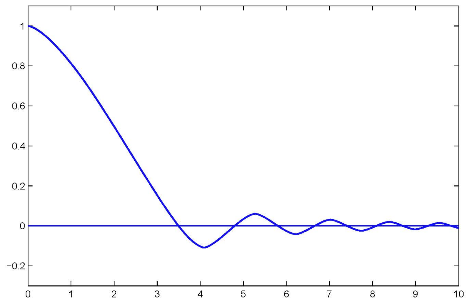
$$|g'(r)|^{p-2}g'(r) = -\beta r g^{\frac{1}{m}}(r) - \frac{\alpha - \beta N}{r^{N-1}} \int_0^r s^{N-1} g^{\frac{1}{m}}(s)ds, \tag{1.59}$$

or, equivalently,

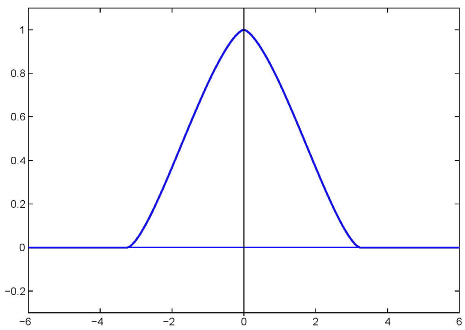
$$|g'(r)|^{p-2}g'(r) + \frac{\beta N - \alpha}{N r^{N-1}} \int_0^r s^N (g^{\frac{1}{m}})'(s)ds = -\frac{\alpha}{N} g^{\frac{1}{m}}(r). \tag{1.60}$$

We will make a formal study of self-similar solutions of the (DNLE) by considering the following cases: $\beta = 0$, $\alpha - \beta N = 0$, $\alpha - \beta N > 0$ and $\alpha - \beta N < 0$. We define the numbers

$$\alpha_B := \frac{1}{m(p-1) - 1 + (p/N)}, \ \beta_B := \frac{\alpha_B}{N} = \frac{1}{(m(p-1)-1)N + p}, \ \alpha_0 := \frac{1}{m(p-1)-1}.$$



(a) Case $\beta = 0, \alpha = \alpha_0$



(b) Case $\alpha = \beta N$

FIGURE 1.2: Self-similar solutions of the (DNLE)

I. Case $\beta = 0,\ \alpha = \alpha_0$. Separate variables solutions

They have the form

$$U(x,t) = (t+s)^{\mu} f(x),$$

where $s > 0$ is a constant and f is the solution of the stationary problem (1.19). Notice that the functions belonging to this family are self-similar solutions according to the previous definition when f is a radial function. These functions are very useful since they indicate the asymptotic behaviour of u, the general solution of the Dirichlet problem (DNLE-d) and thus we will use them for comparison. A second aspect is that they do not propagate and thus we have to consider also different types of self-similar solutions.

II. Case $\alpha = \beta N$. Barenblatt solutions

For more details we refer to [105]. A Barenblatt solution (also called source type solution) exists for the (DNLE) in the "good range"

$$m(p-1) + \frac{p}{N} > 1$$

that includes of course $m \geq 1,\ p = 2$ (the degenerate PME) and $m = 1,\ p \geq 2$ (the degenerate PLE). When, moreover, $m(p-1) > 1$ (that we consider in Sections 1.2-1.3), Barenblatt solutions have the form:

$$\mathcal{U}(x,t;a,s) = c(t+s)^{-\alpha}\left(a^{\frac{p}{p-1}} - |x(t+s)^{-\beta}|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{m(p-1)-1}},$$

where $s > 0$ is a positive parameter and

$$\alpha = \alpha_B = \frac{1}{m(p-1) - 1 + (p/N)}, \quad \beta = \beta_B = \frac{\alpha_B}{N},$$
$$c = \left(\frac{m(p-1)-1}{p}\left(\frac{\alpha}{N}\right)^{\frac{1}{p-1}}\right)^{\frac{p-1}{m(p-1)-1}}. \tag{1.61}$$

When $s = 0$, this function has a multiple of the Dirac delta as initial trace

$$\lim_{t\to 0} \mathcal{U}(x,t) = M\delta_0(x).$$

The remaining parameter $a > 0$ is free and can be uniquely determined in terms of the initial mass $\displaystyle\int_{\mathbb{R}^N} \mathcal{U}dx = M$.

Barenblatt solutions $\mathcal{U}$ are compactly supported and they propagate with finite speed. We will use them as a lower bound in order to prove the positivity of u inside Ω. Since $\mathcal{U}^m$ has a flat landing contact (zero derivative at the boundary of the support) we can not obtain a quantitative lower bound for u up to the boundary of Ω. Their advantage is that they have an explicit formula which is very advantageous for computations.

III. Case $\alpha > \beta N$. Intermediate self-similar solutions and subsolutions

In this case

$$\alpha > \alpha_B, \quad 0 < \beta < \beta_B.$$

This family of self-similar solutions, which we denote by $\mathcal{V}$, inherits some useful properties of the Barenblatt solutions and the separate variables solutions: $\mathcal{V}$ has a compact support that propagates and $g = h^m$ has a transversal cross through the r axis. This is explained as follows. Consider $[0, a)$ the largest interval starting from 0 where $h > 0$. Then, from (1.59) we obtain that $g'(r) < 0$ for $r \in [0, a)$ and thus

$$|g'(r)|^{p-2}g'(r) \leq -\beta r g^{\frac{1}{m}}(r).$$

Furthermore, this implies that

$$-g'(r) \geq \beta^{\frac{1}{p-1}} r^{\frac{1}{p-1}} g^{\frac{1}{m(p-1)}}.$$

Integrating from 0 to r with $g(0) = M^m$ we obtain that

$$g(r) \leq \left(M^{\frac{m(p-1)-1}{p-1}} - \frac{m(p-1)-1}{mp} \beta^{\frac{1}{p-1}} r^{\frac{p}{p-1}} \right)^{\frac{m(p-1)}{m(p-1)-1}},$$

for all $r \in (0, a)$. From this upper bound we derive an estimate for the point a where $h(a) = 0$

$$a \leq M^{\frac{m(p-1)-1}{p}} \left(\frac{mp}{m(p-1)-1} \right)^{\frac{p-1}{p}} \beta^{-\frac{1}{p}}.$$

The important fact is that a is finite, thus $\mathcal{V}$ has a transversal cross through the r axis at the point $r = a$ with

$$g'(a) = -\left(\frac{\alpha - \beta N}{a^{N-1}} \int_0^a s^{N-1} g^{1/m}(s) ds \right)^{\frac{1}{p-1}} =: k_0 < 0. \tag{1.62}$$

We will use the form

$$\mathcal{V}(x, t; M, s) = (t + s)^{-\alpha} [h(r; M)]_+,$$

and therefore the following characterization of the support

$$\text{supp } \mathcal{V}(x, t; M, s) = \{(x, t) : |x| \leq a(t + s)^{\beta}, t \geq 0\}.$$

We denote by

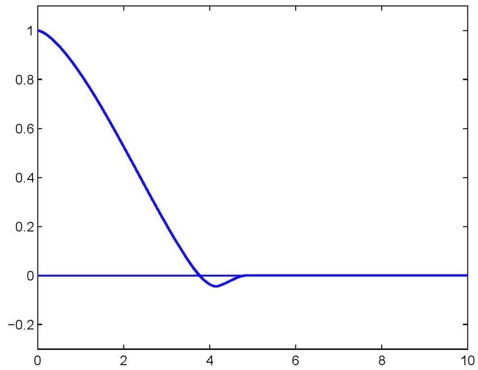
$$a = a(M), \quad k = k(M), \quad h(\cdot) = h(\cdot; M)$$

in order to emphasize their correspondence to the Cauchy problem with initial conditions $h(0) = M, \; h'(0) = 0$. We remark that

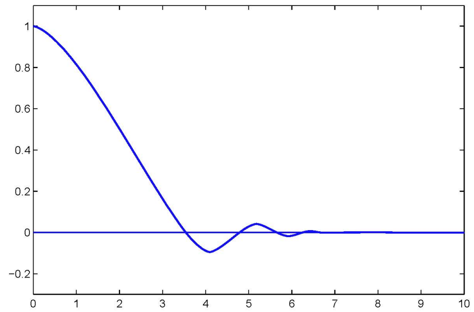
$$h(a(M); M) = 0, \quad h(r; M) = M h(M^{-\frac{m(p-1)-1}{p}} r; 1) \quad \text{and } a(M) = M^{\frac{m(p-1)-1}{p}} a(1). \tag{1.63}$$

Subsequently we will consider the self-similar solutions described above only on $[0, a)$, the largest interval starting from 0 where they are positive; for complete definition, on $[a, \infty)$ they are assigned zero values. This way, they are sub-solutions of the (DNLE-d).

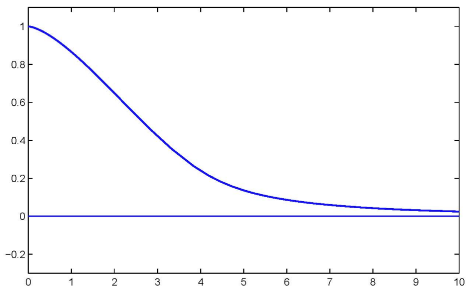
Remark. In the present work we do not study the behaviour of these functions when g takes negative values. Depending on the values of α and β, g can behave differently, as we can see in Figure 3.



(a) Case $\alpha > \beta N$



(b) Case $\alpha > \beta N$



(c) Case $\alpha < \beta N$

FIGURE 1.3: Self-similar solutions of the (DNLE)

IV. Case $\alpha < \beta N$

The self-similar function corresponding to the profile g in this case does not have a compact support, hence this class is not useful for our estimates. For completeness we will provide a formal characterization of these functions.

Recall that in this case

$$0 < \alpha < \alpha_B, \quad \beta > \beta_B.$$

Using basic computations as in the previous case, one can easily prove that g is a positive decreasing function converging to 0 when r goes to ∞. Moreover, for every $r_0 > 0$ there exists $C = C(r_0) > 0$ such that

$$g(r) \geq Cr^{-\frac{\alpha m}{\beta}}, \quad \forall r \geq r_0. \tag{1.64}$$

Asymptotic decay. We point out that g behaves as $r \to \infty$ like $G_a(r) = ar^{-\gamma}$, $\gamma = \alpha m/\beta$. We continue with a formal proof.
We will consider the following series expansions of g:

$$g(r) = a_1 r^{-\gamma} + ...$$

as r goes to ∞, where $\gamma > 0$ is an exponent to be determined and "..." is representative for lower order terms. Then

$$\alpha h(r) + \beta r h'(r) = a_1^{\frac{1}{m}} \left(\alpha - \beta\frac{\gamma}{m}\right) r^{-\frac{\gamma}{m}} + ...,$$
$$\frac{1}{r^{N-1}} \left(r^{N-1}|g'(r)|^{p-2}g'(r)\right)' = a_1^{p-1}\gamma^{p-1}(\gamma(p-1)+p-N)r^{-(\gamma+1)(p-1)-1} + ...$$

For a comparison of the first terms we notice that

$$-(\gamma+1)(p-1) - 1 < -\frac{\gamma}{m}.$$

Thus *the leading asymptotic term* in the expansion of the ODE formula (1.57) is

$$a_1^{\frac{1}{m}} \left(\alpha - \beta\frac{\gamma}{m}\right) r^{-\frac{\gamma}{m}}.$$

Moreover, we notice that the coefficient $\alpha - \beta\gamma/m = 0$, from where we deduce that the *exponent of the leading asymptotic term* is

$$\gamma = \frac{\alpha m}{\beta}. \tag{1.65}$$

At this time we have no information about a_1. The coefficient of the remaining term is

$$a_1^{p-1}\gamma^{p-1}(\gamma(p-1)+p-N),$$

whose sign depends on the values of β.

Let

$$\beta_B := \frac{1}{(m(p-1)-1+p/N)\,N}, \quad \alpha_B := \frac{1}{m(p-1)-1+p/N},$$
$$\gamma_1 := \frac{N-p}{p-1}, \quad \beta_1 := \frac{m(p-1)}{(m(p-1)-1+p/N)\,N} = m(p-1)\beta_B, \quad \alpha_1 := \frac{1-p\beta_1}{m(p-1)-1}.$$

The sign of coefficient $\gamma(p-1)+p-N$ can be now obtained depending on the values of β.

1. Case $\gamma(p-1)+p-N>0 \Leftrightarrow \begin{cases} \gamma>\gamma_1,\ \beta<\beta_1,\ \alpha>\alpha_1; \\ \text{or } p\geq N. \end{cases}$

2. Case $\gamma(p-1)+p-N<0 \Leftrightarrow \begin{cases} \beta\in(\beta_1,1/p),\ \alpha<\alpha_1, & p<N; \\ \text{impossible}, & p\geq N. \end{cases}$

3. Case $\gamma(p-1)+p-N=0 \ \Leftrightarrow\ \gamma=\gamma_1,\ \beta=\beta_1,\ \alpha=\alpha_1$. Then $g(r)=r^{-\gamma}$ is the solution of the ODE (1.57).

We can observe that in the first two cases we can not deduce the decay of g and we need to perform a second approximation.

1.5 The quasilinear case $m(p-1)=1$

We study the large-time asymptotic behaviour of solutions of the problem (DNLE-d) in the quasilinear case $m(p-1)=1$. The study of the asymptotic behaviour in the present case $m(p-1)=1$ differs considerably from the case $m(p-1)>1$ previously studied for several reasons. Firstly, the proof in the case $m(p-1)>1$ is based on monotonicity: the rescaled solution $t^{\mu}u(x,t)\nearrow v(x,t)$. This argument cannot be applied in the present case. Besides, we have no universal a-priori estimates similar to the degenerate case.

As usual, the problem is better understood via the method of rescaling. We consider

$$v(x,t)=e^{\lambda t}u(x,t),\quad t\in[0,\infty),x\in\Omega, \tag{1.66}$$

where λ is a real parameter whose choice we will justify in the next subsection. Then v is a solution of the *rescaled problem*

$$\begin{cases} v_t(x,t)=\Delta_p v^m(x,t)+\lambda v(x,t) & \text{for } t>0 \text{ and } x\in\Omega, \\ v(x,0)=u_0(x) & \text{for } x\in\Omega, \\ v(x,t)=0 & \text{for } t>0 \text{ and } x\in\partial\Omega. \end{cases} \tag{1.67}$$

1.5.1 The associated stationary problem

Consider the stationary problem associated to Problem (1.67):

$$\Delta_p f^m+\lambda f=0 \quad \text{in } \Omega, \qquad f(x)=0 \quad \text{for } x\in\partial\Omega, \tag{1.68}$$

that can be rewritten in terms of $z=f^m$ as

$$\Delta_p z+\lambda|z|^{p-2}z=0 \quad \text{in } \Omega, \qquad z=0 \quad \text{on } \partial\Omega. \tag{1.69}$$

We want to obtain solutions $z(x)>0$ in Ω. We call *eigenvalues* the λ-s for which there exists a nontrivial solution of Problem (1.69), which is known as the eigenvalue problem for the $p-$Laplacian.

The first eigenvalue λ_1 of Problem 1.69 is defined by

$$\lambda_1 := \inf \left\{ \int_\Omega \frac{|\nabla \varphi|^p}{|\varphi|^p}, \ \varphi \in W_0^{1,p}(\Omega) \right\}, \tag{1.70}$$

that is $\lambda_1 = \mathcal{C}^{-p}$ where $\mathcal{C}$ is the best constant of the embedding $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$. The following result was proved in [6, 85].

Theorem 1.5.1. ***(Simplicity and isolation of the first eigenvalue of Problem** 1.69)* *The first eigenvalue λ_1 of Problem 1.69 is simple and isolated. Moreover, λ_1 is the unique positive eigenvalue of Problem 1.69 having a nonnegative eigenfunction.*

1.5.2 Preliminary estimates for the evolution problem

We state two results obtained by Manfredi and Vespri (Theorems 1.4 and 1.4 from [86]).

Theorem 1.5.2. *Consider $m(p-1) = 1$. Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain. Then there exists a unique solution of the problem* (DNLE-d) *corresponding to an initial datum $u_0 \in L^1(\Omega)$. Moreover, for all $t \geq 1$, there exists a constant $c(t)$ such that*

$$|u(x,t)| \leq \gamma_1 e^{-\lambda_1 t} c(t) f(x), \quad \forall x \in \Omega, \tag{1.71}$$

where f is a solution of the problem (1.68) *with $\lambda = \lambda_1$ such that $f \in C^0(\overline{\Omega})$, $f^{\frac{p-2}{p-1}} \nabla f \in L^p(\Omega)$ and γ_1 is a positive constant depending only on the data N, p, m, the L^1 norm of u_0 and the $C^{1,\alpha}$ norm of $\partial\Omega$.*

Theorem 1.5.3. *Consider the hypothesis of the previous theorem and assume moreover that $u_0 \geq 0$ and not identically zero. Then, for every $t \geq 1$, there exist constants $\underline{c}(t)$, $\overline{c}(t) \in \mathbb{R}_+$ such that the following estimate holds*

$$e^{-\lambda_1 t} \underline{c}(t) f(x) \leq u(x,t) \leq e^{-\lambda_1 t} \overline{c}(t) f(x), \quad \forall x \in \Omega. \tag{1.72}$$

Moreover, we also have

$$\gamma_1(t) e^{-\lambda_1 t} d(x)^{p-1} \leq u(x,t) \leq \gamma_2(t) e^{-\lambda_1 t} d(x)^{p-1}, \quad \forall x \in \Omega, t > 1 \tag{1.73}$$

and

$$|\nabla u(x,t)| \leq \gamma_3(t) e^{-\lambda_1 t} d(x)^{p-2}, \quad \forall x \in \Omega, t > 1. \tag{1.74}$$

Remarks

- **Reduction.** The estimates given by the previous theorems are true for every $t \geq t_0$, where $t_0 > 0$ is fixed. Since the doubly nonlinear equation is invariant under a time displacement, we can assume that the previous estimates are valid for every $t \geq 0$, otherwise we can start with initial data $u(t_0)$. We assume therefore such a displacement in time has been done.

- We can fix f in any way, up to a multiplicative constant. Therefore we fix f a nonnegative solution of the problem (1.68).

Inspired by the ideas from [74], we will obtain more information about the constants $\overline{c}(t)$ and $\underline{c}(t)$. Let us define

$$\overline{c}(t) = \inf\{c : v(x,t) \le cf(x)\}, \quad \underline{c}(t) = \sup\{c : v(x,t) \ge cf(x)\}. \tag{1.75}$$

According to Theorem 1.5.3, the previous definition makes sense: $\overline{c}(t) < \infty$ and $\underline{c}(t) > 0$ for every $t \ge 0$. Thus, we can take $\underline{c}(t)$ and $\overline{c}(t)$ to be the best constant such that estimate (1.72) holds. It is a simple consequence of the Maximum principle that $\overline{c}(t)$ is a decreasing function of t and $\underline{c}(t)$ increasing function of t. Therefore the following limits exist:

$$\overline{c}_\infty = \lim_{t\to\infty} \overline{c}(t), \quad \overline{c}(t) \searrow \overline{c}_\infty, \tag{1.76}$$

$$\underline{c}_\infty = \lim_{t\to\infty} \underline{c}(t), \quad \underline{c}(t) \nearrow \underline{c}_\infty. \tag{1.77}$$

In addition, the constants $\underline{c}(t)$ and $\overline{c}(t)$ are uniformly bounded

$$C_0 \le \underline{c}(t) \le \underline{c}_\infty \le \overline{c}_\infty \le \overline{c}(t) \le C_1, \quad \forall t \ge 0. \tag{1.78}$$

We can sum up what we have proved so far in the following lemma.

Lemma 1.5.1. *Let v be a solution of the rescaled problem* (1.66). *Then there exist the positive constants $C_0,\ C_1,\ C_2 > 0$ such that*

$$C_0 f(x) \le v(x,t) \le C_1 f(x), \quad \forall x \in \Omega, t \ge 0, \tag{1.79}$$

$$C_0 d(x) \le v(x,t) \le C_1 d(x), \quad \forall x \in \Omega, t \ge 0, \tag{1.80}$$

and

$$|\nabla v(x,t)| \le C_2 d(x)^{p-2}, \quad \forall x \in \Omega, t \ge 0, \tag{1.81}$$

where $C_0, C_1, C_2 > 0$ depend on Ω, λ_1 and f is the positive solution of problem (1.68) *we have taken.*

As a consequence we obtain the uniform convergence, up to subsequences, of $v(t,\cdot)$ to a stationary profile.

Theorem 1.5.4. ***(Uniform convergence to an asymptotic profile up to subsequences)***
Consider $m(p-1) = 1$. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$. Let $u(t,\cdot)$ be a weak solution to Problem (DNLE-d) *corresponding to the nonnegative initial datum $u_0 \in L^1(\Omega)$. Then for any given $T > 0$ there exists a sequence $\tau_n \to \infty$ such that*

$$|e^{\lambda_1(\tau_n+t)} u(\tau_n + t, s) - c_* f(x)| \to 0, \quad \tau_n \to \infty, \tag{1.82}$$

uniformly for $x \in \Omega$ and $0 \le t \le T$, where f is the positive solution of problem (1.68) *we have taken and c_* is a positive constant.*

Proof. **I. Energy estimates**

Ia. We consider the following energy functional

$$E(t) = E[v(t)] := \frac{1}{p}\int_\Omega |\nabla v^m(x,t)|^p dx - \frac{m}{m+1}\lambda_1 \int_\Omega v^{m+1}(x,t)dx.$$

We compute the energy dissipation

$$-\frac{d}{dt}E(t) = I(t) = m\int_\Omega v^{m-1}v_t^2 dx \ge 0,$$

which means that $E(t)$ is a non-increasing function and

$$E(t_1) - E(t_2) = \int_{t_1}^{t_2} I(t)dt.$$

As well, we deduce that the integral

$$\int_{t_1}^{t} I(t)dt$$

is convergent as $t \to \infty$ and $E(t)$ has a limit as $t \to \infty$.

Since $v(x,t)$ is bounded in Ω uniformly for $t \ge 0$ we obtain that

$$\int_\Omega |\nabla v^m(x,t)|^p dx \le M, \quad \forall t \ge 0, \tag{1.83}$$

in other words $|\nabla v^m(t,\cdot)|$ is uniformly bounded in $L^p(\Omega)$ for $t \ge 0$.

Ib. As a consequence of (1.83) one can prove via Hölder's Inequality the following technical result

$$\int_\Omega (\Delta_p v^m(x,t_1))v^m(x,t_2)dx \le M, \quad \forall t_1, t_2 \ge 0. \tag{1.84}$$

We have

$$\begin{aligned}
\left|\int_\Omega (\Delta_p v^m(x,t_1))v^m(x,t_2)dx\right| &= \left|\int_\Omega |\nabla v^m(t_1)|^{p-2}\nabla v^m(t_1)\nabla v^m(t_2)dx\right| \\
&\le \left(\int_\Omega \left(|\nabla v^m(t_1)|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla v^m(t_2)|^p\right)^{\frac{1}{p}} \\
&= \left(\int_\Omega |\nabla v^m(t_1)|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla v^m(t_2)|^p\right)^{\frac{1}{p}} \\
&\le M^{\frac{p-1}{p}+\frac{1}{p}} = M, \quad t_1, t_2 \ge t_0.
\end{aligned}$$

II. Convergence. We define

$$\tilde{v}_\tau(x,t) = v(x,t+\tau), \quad t,\tau > 0,\ x \in \Omega. \tag{1.85}$$

Then $\tilde{v}_\tau$ is still a solution of Problem (1.67) with initial data $\tilde{v}_\tau(x,0) = v(x,\tau)$.

We fix $T > 0$. The family $(\tilde{v}_\tau)$ is relatively compact in

$$X = L^\infty(\overline{\Omega} \times [0,T])$$

thus it converges along subsequences

$$v_{\tau_n}(x,t) \to S(x,t) \qquad \text{uniformly in } \ (x,t) \in \Omega \times [0,T].$$

From the a-priori estimates we deduce the boundedness of v

$$\tilde{C}_0 \le v(x,t) \le \tilde{C}_1, \quad \forall t \ge 0,\ x \in \Omega,$$

and since S is the limit of $v_{\tau_n}(x,t)$, then S also satisfies the same lower and upper bounds.

In what follows we fix such a subsequence (τ_n) and the corresponding limit $S(x,t)$.

III. Convergence in measure of gradients

Similar to the case $m(p-1) > 1$ one can prove the convergence in measure of the sequence $(\nabla v^m_{\tau_n}(\cdot,\cdot))_n$, where in the present case

$$v_{\tau_n} : \Omega \times [0,T] \to [0,\infty).$$

More exactly, we can prove by similar methods that the sequence $(\nabla v^m_{\tau_n})_{n>0}$ is Cauchy in measure, thus it converges in measure to a function $W : \Omega \times [0,T] \to \mathbb{R}^N$. It is a well known fact (Lemma 1.6.1 in the Appendix) that if a sequence is uniformly bounded in L^p and converges in measure, then it converges strongly in any L^q, for any $1 \le q < p$. It follows that

$$\nabla v^m_{\tau_n} \to W \quad \text{ strongly in } (L^q(\Omega \times [0,T]))^N \text{ when } \tau_n \to \infty, \text{ for every } 1 \le q < p.$$

Thus, we get that, up to subsequences,

$$\nabla v^m_{\tau_n}(\cdot,\cdot) \to W(\cdot,\cdot) \quad \text{a.e. in } \Omega \times [0,T], \tag{1.86}$$

and we conclude that

$$W(x,t) = \nabla S^m(x,t).$$

IV. The limit is a solution of the stationary problem

We multiply equation (1.67) by any test function $\phi(x) \in C_c^\infty(\Omega)$ and integrate in space, $x \in \Omega$, and time between τ_n and $\tau_n + T$. We get that

$$\int_\Omega (v(\tau_n+T) - v(\tau_n))\phi dx = -\int_{\tau_n}^{\tau_n+T} \int_\Omega |\nabla v^m|^{p-2} \nabla v^m \nabla \phi dx dt + \lambda_1 \int_{\tau_n}^{\tau_n+T} \int_\Omega v\phi dx dt. \tag{1.87}$$

(i) The left hand side term of (1.87) is uniformly bounded independently of T:

$$\left| \int_\Omega v(\tau_n+T)\phi dx - \int_\Omega v(\tau_n)\phi dx \right| \leq 2\tilde{C}_1 |\Omega| \|\phi\|_{L^\infty(\Omega)}. \tag{1.88}$$

Furthermore, if ϕ is supported in a compact $K \subset \Omega$ where $0 < c_1 \leq v \leq c_2$ and $0 < s \leq T$, then

$$\begin{aligned} &\left| \int_\Omega (v(\tau_n+s) - v(\tau_n))\,\phi dx \right| \leq \int_{\tau_n}^{\tau_n+s} \int_\Omega |v_t(t)\,\phi| dx dt \leq \\ &\leq CT^{1/2} \left(\int_{\tau_n}^{\tau_n+s} \int_\Omega ((v^{(m+1)/2})_t)^2 dx dt \right)^{1/2} = CT^{1/2} \left(\int_{t_n}^{\tau_n+s} I(t) dt \right)^{1/2}. \end{aligned} \tag{1.89}$$

Since the double integral $\int_1^\infty I(t)$ is finite, the integral on the right hand side goes to zero as $\tau_n \to \infty$. On the other hand,

$$\begin{aligned} \int_\Omega v(\tau_n+s)\phi dx - \int_\Omega v(\tau_n)\phi dx &= \int_\Omega v_{\tau_n}(x,s)\phi dx - \int_\Omega v_{\tau_n}(x,0)\phi dx \\ &\to \int_\Omega S(x,s)\phi dx - \int_\Omega S(x,0)\phi dx, \quad \tau_n \to \infty. \end{aligned}$$

Therefore, we showed that

$$\int_\Omega (S(x,s) - S(x,0))\,\phi\,dx = 0,$$

for any $0 < s \leq T$ and any test function ϕ with supp $\phi = K \subset\subset \Omega$. Since K is arbitrary, the result holds for all $\phi \in C_c^\infty(\Omega)$ which implies that S is independent of time on $[0,T]$:

$$S(s) = S(0), \quad \forall s \in [0,T]. \tag{1.90}$$

(ii) For the right hand side, we continue as follows. Let $\tau_n \to \infty$. Since $\nabla v_{\tau_n}^m \to W = \nabla S^m$ a.e. in $\Omega \times [0,T]$ then using Lemma 1.6.2 of the Appendix we get that

$$\int_{\tau_n}^{\tau_n+T} \int_\Omega |\nabla v^m|^{p-2} \nabla v^m \nabla \phi dx dt = \int_0^T \int_\Omega |\nabla v_{\tau_n}^m|^{p-2} \nabla v_{\tau_n}^m \nabla \phi dx dt$$

$$\to T \int_\Omega |\nabla S^m|^{p-2} \nabla S^m \nabla \phi dx.$$

The last integral

$$\int_{\tau_n}^{\tau_n+T}\int_\Omega v\phi dxdt = \int_0^T\int_\Omega v_{\tau_n}(x,t)\phi(x)dxdt \to T\int_\Omega S(x,t)\phi(x)dx, \quad \tau_n \to \infty.$$

Therefore

$$-\int_\Omega |\nabla S^m|^{p-2}\nabla S^m \nabla\phi dx + \lambda_1 \int_\Omega S\phi dxdt = 0, \quad \forall t \in [0,T], \tag{1.91}$$

and thus S is a weak solution of the stationary problem (1.68). According to the Theorem 1.5.1,

$$S(x,t) = S(x,0) = c_* f(x), \quad \forall t \in [0,T],\ x \in \Omega.$$

For simplicity, we denote $S(x) := c_* f(x)$.

□

Remarks

• $V = S^m$, where $S := c_* f$, is a positive solution of the eigenvalue problem for the $p-$Laplacian:

$$\Delta_p V + \lambda_1 V^{p-1} = 0 \text{ in } \Omega, \quad V = 0 \text{ on } \partial\Omega. \tag{1.92}$$

• As a consequence of (1.76) and (1.82) the constants $\underline{c}_\infty$, $\overline{c}_\infty$ and c_* satisfy

$$\underline{c}_\infty \le c_* \le \overline{c}_\infty.$$

• The previous proposition does not guarantee the uniqueness of a stationary limit S. Therefore the rescaled solution $v(x,t)$ may oscillate between the bounds $\underline{c}_\infty f(x)$ and $\overline{c}_\infty f(x)$ by converging on subsequences to asymptotic profiles of the form $c_* f$ with $\underline{c}_\infty \le c_* \le \overline{c}_\infty$. This kind of behaviour has to be considered in the case of some parabolic evolution equations, for example in the case of signed solutions of the porous medium equation $u_t = \Delta u^m$, $m > 1$. The set of possible asymptotic profiles is obtained as the ω-limit of the solution and it is contained in the set of classical solutions of the associated stationary (elliptic) problem (we refer to the survey [104]).

Next, we will prove that an oscillating behaviour is not possible. More, exactly, we show that

$$\underline{c}_\infty = c_* = \overline{c}_\infty,$$

which guarantees the existence of a unique asymptotic profile $S = c_* f$ and therefore the uniform convergence of the rescaled solution $v(x,t)$ to S for all times, uniformly in $\overline{\Omega}$.

To this aim, we will study the behaviour of the quotient v/S up to the boundary.

1.5.3 The relative error function and its equation

Assumptions (A). In what follows we make the assumptions: $\Omega \subseteq \mathbb{R}^N$ is a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$, $u(t,\cdot)$ denotes the weak solution to Problem (DNLE-d) in the case $m(p-1)=1$ corresponding to the nonnegative initial datum $u_0 \in L^1(\Omega)$. We fix $T>0$, the corresponding sequence $\tau_n \to \infty$ and the constant $c_* \in [\underline{c}_\infty, \overline{c}_\infty]$ obtained in Theorem 1.5.4 for which the rescaled solution $v(x,t)=e^{\lambda_1 t}u(x,t)$ converges

$$\|e^{\lambda_1(\tau_n+t)}u(\tau_n+t,\cdot)-c_* f(\cdot)\|_{L^\infty(\Omega)} \to 0, \quad \tau_n \to \infty$$

uniformly for $t \in [0,T]$, where f is the positive solution of problem (1.68) we have taken. We denote $S := c_* f$ and we call it a *possible asymptotic profile.*

Starting from this partial convergence result, we will obtain a much stronger convergence as $t \to \infty$: we show the uniqueness of the asymptotic profile and the convergence in relative error of $v(t,\cdot)$ to $S(\cdot)$ up to the boundary as a consequence of the next proposition and estimates (1.79).

Proposition 1.5.1. ***(Behaviour up to the boundary)*** *Under the assumptions (A) there exists a unique constant $c_*>0$ depending on u_0 and Ω, and for given $\epsilon>0$ there exists $t(\epsilon)>0$ such that*

$$-\epsilon < \frac{v^m(x,t)}{S^m(x)} - 1 < \epsilon, \quad \forall x \in \Omega, \quad \forall t \geq t(\epsilon). \tag{1.93}$$

Moreover, $S=c_* f$ is the solution of the problem (1.68) *announced in Theorem 1.5.4.*

Motivated by the techniques used by Bonforte, Grillo and Vázquez in [26] we will use the so called relative error function and the method of barriers.

To this aim, we introduce the *Relative Error Function*(REF)

$$\phi(x,t) = \frac{v^m(x,t)}{S^m(x)} - 1, \quad v^m = S^m(\phi+1) = V(\phi+1), \quad \text{and} \quad V = S^m. \tag{1.94}$$

Notations. We define

$$\Omega_{I,\delta} = \{x \in \overline{\Omega} : d(x) > \delta\}, \quad \Omega_\delta = \Omega \setminus \overline{\Omega_{I,\delta}} = \{x \in \overline{\Omega} : d(x) < \delta\},$$

where in what follows $\delta>0$ is considered to be a small positive parameter (see Subsection 1.6.3 of the Appendix for properties of the distance to the boundary function).

Properties of the REF

• *The parabolic equation of the REF.* Using the equations satisfied by v and V and relation $m(p-1)=1$ we obtain that

$$(p-1)(1+\phi)^{p-2}\phi_t = V^{-(p-1)}\Delta_p((\phi+1)V) + \lambda_1(\phi+1)^{p-1}. \tag{1.95}$$

• ϕ is uniformly bounded in (x,t) for $t>0$. This can be derived from the estimates (1.80) on v and S, the latter being a stationary solution:

$$\left(\frac{C_0}{C_1}\right)^m - 1 = C_{2,m} \leq \phi \leq C_{3,m} = \left(\frac{C_1}{C_0}\right)^m - 1.$$

• In any interior region $\Omega_{I,\delta} \subset \Omega$, the REF function ϕ satisfies

$$1 + \phi = \frac{v^m}{V} > 0 \quad \text{in } \Omega_{I,\delta} \quad \text{for any } t \geq 0.$$

• *Regularity of solutions of the parabolic equation* (1.95). Since ϕ is also bounded in the interior of Ω, we conclude that the parabolic equation (1.95) is neither degenerate nor singular in the interior of Ω. Also the solution ϕ of such a parabolic equation is Hölder continuous in any inner region $\overline{\Omega}_{I,\delta} \subset \Omega$, since both v and S are Hölder continuous and positive in the interior of Ω.

Convergence of the REF in an interior region of Ω. Under the running assumptions, we know by Theorem 1.5.4 that

$$\sup_{\overline{\Omega}} |v(\tau_n + t) - S| \to 0 \quad \text{as } n \to \infty,$$

uniformly for $t \in [0, T]$ for a fixed $T > 0$ and a corresponding sequence $(\tau_n)_n$, but this is not sufficient to prove the convergence of the quotient v^m/S^m to 1 in the whole Ω, since at the boundary there is the problem caused by the fact that both v and S are 0 and therefore the parabolic equation (1.95) may degenerate at the boundary. However such a problem is avoided in any interior region where both v and S are strictly positive.

We can sum up the results we proved so far in the following lemma.

Lemma 1.5.2. ***(Inner convergence)*** *Let v the solution of the rescaled problem* (1.67) *and S the solution of stationary problem* (1.68) *corresponding to a given $T > 0$ and a sequence $\tau_n \to \infty$ as in Theorem 1.5.4. Let ϕ be the associated relative error function defined by* (1.94). *Then*

$$\|\phi(\tau_n + t, \cdot)\|_{L^\infty(\Omega_{I,\delta})} = \sup_{\Omega_{I,\delta}} |\phi(\tau_n + t, \cdot)| \to 0,$$

as $\tau_n \to \infty$ uniformly in $x \in \Omega_{I,\delta}$ and $0 \leq t \leq T$, for any given $\delta > 0$.

1.5.4 Construction of the upper barrier and consequences

In order to finish the proof of Theorem 1.1.2 we have to prove the uniform convergence of ϕ up to the boundary. This will be realized using a barrier argument based on the ideas from [26].

Let us first point out some connections between the distance to the boundary function and the solutions of the eigenvalue problem (1.92).

Lemma 1.5.3. ***(Properties of the asymptotic profile** $V = S^m$) *Let V be a solution of the eigenvalue problem* (1.92). *Then V satisfies the following estimates:*

1. *There exist $C_0 > 0$ and $C_1 > 0$ such that*

$$C_0^m d(x) \leq V(x) \leq C_1^m d(x), \quad \forall x \in \Omega.$$

2. *For every* $0 < \xi_1 \leq \xi_0$ *there exists a constant* $\beta_0 > 0$ *such that*

$$\nabla V(x) \cdot \nabla d(x) \geq \beta_0 > 0, \quad \forall x \in \Omega_{\xi_1}.$$

3. *For every* $0 < \xi_1 \leq \xi_0$ *there exists a constant* $K_1 > 0$ *such that*

$$0 < K_1 \leq |\nabla V| \leq K_2, \quad \forall x \in \Omega_{\xi_1}. \tag{1.96}$$

Proof. Point 1. is a consequence of Lemma 1.5.1.

The proof of the point 2. is similar to the one given in [26], since the function V involved has the same properties as its correspondent in the fast diffusion problem.

The function $d(x)$ satisfies $0 < c \leq |\nabla d(x)| \leq 1$ for all $x \in \Omega_{\xi_0}$, for a constant c (in Lemma 1.6.3 of the Appendix A we will give a list of the properties of the function $d(x)$). As a consequence of this property, the result proved in point 1. above and estimates (1.81) we conclude the proof of point 3., since

$$|\nabla V| = |mS^{m-1}\nabla S| \leq K_2 d(x)^{p-2+(m-1)/m} = K_2 d(x)^{m(p-1)-1} = K_2, \quad \forall x \in \Omega.$$

□

Next, we present the construction of the barrier that plays an important role in the estimate of REF ϕ close to the boundary. We mention that our construction is different from the one of [26], where the operator was the usual Laplacian Δ. In our case, the p-Laplacian operator contains also mixed derivatives of second order whose estimate is more technical.

Lemma 1.5.4. ***(Upper barrier)*** *We can choose positive constants* A, B, C *so that for every* $t_0 > 0$ *the function*

$$\Phi(x,t) = C - BV(x) - A(t - t_0), \tag{1.97}$$

is a super-solution to equation (1.95) *on a parabolic region near the boundary*

$$\Sigma_\Phi = \{(x,t) \in (t_0, \infty) : \Phi(x,t) \geq -1\},$$

and moreover $\Sigma_\Phi \subset \Omega_{\xi_1} \times (t_0, T_0)$, *where* $\xi_1 \leq \xi_0$.

Proof. We will prove that the function (1.97) is a supersolution for the equation (1.95) on the parabolic region Σ_Φ if we can find constants A, B and C such that

$$(p-1)(1+\Phi)^{p-2}\Phi_t \geq V^{-(p-1)}\Delta_p((\Phi+1)V) + \lambda_1(\Phi+1)^{p-1}. \tag{1.98}$$

We will prove that a convenient choice for A, B and C will be of the form

$$\left(\lambda_1(C+1) + A(p-1)\right)\xi_1^{p-1} \leq \omega B, \tag{1.99}$$

where

$$\omega = \min\{1, 2^{2-p}\} \cdot \frac{2(p-1)K_1^p}{C_1}.$$

From the beginning we assume that $(x,t) \in \Sigma_\Phi \subset \Omega_{\xi_1} \times (t_0, T_0)$ where T_0 is such that

$$0 < T_0 - t_0 \leq \frac{C - BV(x)}{A}.$$

The left hand side term satisfies

$$(p-1)(1+\Phi)^{p-2}\Phi_t = -(p-1)A(1+\Phi)^{p-2}.$$

The right hand side term is of the form

$$V^{-(p-1)}\Delta_p((\Phi+1)V) + \lambda_1(\Phi+1)^{p-1} = V^{-(p-1)}\Delta_p\varphi(V) + \lambda_1(\Phi+1)^{p-1}$$

where

$$\varphi(z) = (C + 1 - Bz - A(t - t_0))z.$$

The term $\Delta_p f$ can be computed as

$$\Delta_p\varphi(V) = |\varphi'(V)|^{p-2}\varphi'(V)\Delta_p V + (p-1)|\varphi'(V)|^{p-2}\varphi''(V)|\nabla V|^p.$$

Properties of the function f

1. Function φ is a concave parabola with zero values at the points $z = 0$ and $z = z_0$ where

$$z_0 := \frac{C + 1 - A(t - t_0)}{B}.$$

2. The derivatives are

$$\varphi'(z) = C + 1 - 2Bz - A(t - t_0), \quad \varphi''(z) = -2B.$$

3. When applied to V, the derivative

$$\varphi'(V) = \Phi + 1 - BV.$$

Moreover, sufficiently close to the boundary, $\varphi'(V)$ is positive and bounded. By choosing

$$\xi_1 = \min\left\{\xi_0,\ \frac{1}{C_1^m}\frac{z_0}{4} = \frac{C + 1 - A(t - t_0)}{4BC_1^m}\right\}, \tag{1.100}$$

we obtain the following bound on Ω_{ξ_1}:

$$0 < V(x) \leq C_1^m d(x) \leq C_1^m \xi_1 \leq \frac{z_0}{4}$$

and then

$$0 < \frac{k_1}{2} \leq f'(V(x)) \leq k_1,$$

where

$$k_1 := \varphi'(0) = C + 1 - A(t - t_0), \qquad \frac{k_1}{2} = \varphi'\left(\frac{z_0}{4}\right) = \frac{1}{2}(C + 1 - A(t - t_0)) > 0. \tag{1.101}$$

Since $p > 1$, we obtain a lower bound for $\varphi'(V)^{p-2}$ on Ω_{ξ_1} as follows

$$\varphi'(V)^{p-2} \geq \beta k_1^{p-2}, \qquad \beta := \min\{1, 2^{2-p}\}. \tag{1.102}$$

Sufficient conditions for the parameters

Since V is a solution of the stationary problem (1.92) then $-\Delta_p V = \lambda_1 V^{p-1}$, and inequality (1.98) can be rewritten as

$$\begin{aligned} &-A(p-1)(1+\Phi)^{p-2} \geq \\ &\qquad - \lambda_1 |f'(V)|^{p-2} f'(V) - 2B(p-1)V^{-(p-1)}|f'(V)|^{p-2}|\nabla V|^p + \lambda_1(\Phi+1)^{p-1}. \end{aligned}$$

The idea is that $V^{-(p-1)}$ can have large values close to the boundary, thus it is sufficient to find A, B and C such that

$$\lambda_1(\Phi+1)^{p-1} + A(p-1)(1+\Phi)^{p-2} \leq 2B(p-1)V^{-(p-1)}|\nabla V|^p|f'(V)|^{p-2}. \tag{1.103}$$

For ξ_1 as in (1.100) and the bounds (1.96), (1.102), the right hand side term of (1.103) satisfies the lower bound

$$2B(p-1)V^{-(p-1)}|\nabla V|^p|f'(V)|^{p-2} \geq 2B(p-1)\beta K_1^p k_1^{p-2}\left(C_1 \xi_1^{p-1}\right)^{-1} =: II.$$

For the left hand side term of (1.103) on Σ_Φ we obtain the upper bound

$$\begin{aligned} &\lambda_1(\Phi+1)^{p-1} + A(p-1)(1+\Phi)^{p-2} \leq \\ &\qquad \lambda_1(C+1-A(t-t_0))^{p-1} + A(p-1)(C+1-A(t-t_0))^{p-2} := I. \end{aligned}$$

Thus it is sufficient to take A, B and C such that

$$\begin{aligned} &(C+1-A(t-t_0))^{p-2}(\lambda_1(C+1-A(t-t_0)) + A(p-1)) \leq \\ &\qquad\qquad 2B(p-1)\beta K_1^p k_1^{p-2}\left(C_1 \xi_1^{p-1}\right)^{-1}. \end{aligned}$$

According to (1.101) this inequality becomes

$$\lambda_1(C+1-A(t-t_0)) + A(p-1) \leq 2(p-1)\beta K_1^p C_1^{-1}\frac{B}{\xi_1^{p-1}}.$$

One can see that a sufficient condition on A, B and C would be

$$\lambda_1(C+1) + A(p-1) \leq 2(p-1)\beta K_1^p C_1^{-1}\frac{B}{\xi_1^{p-1}}.$$

□

We will obtain an upper bound of the REF ϕ at a certain time T_1 up to the boundary as a consequence of comparison of ϕ with the barrier function of Lemma 1.5.4.

Lemma 1.5.5. *Let Φ be the barrier function introduced in Lemma 1.5.4, given by*

$$\Phi(x,t) = C - BV(x) - At,$$

Let $\tau_n \to \infty$ be a sequence along which the REF converges to 1 as stated above. Then for every $\epsilon > 0$ we can choose $n_\epsilon > 0$ and positive constants A, B, C and δ as in Lemma 1.5.4 such that

$$\phi(x, t+\tau_n) \leq \Phi(x,t), \quad \forall x \in \Omega_\delta,\ \forall n \geq n_\epsilon,\ \forall t \in [0, T_1], \tag{1.104}$$

where

$$T_1 = T_1(\epsilon, \delta) = \frac{C - BC_1^m \delta - \epsilon}{A}. \tag{1.105}$$

Proof. We fix $\epsilon > 0$ and consider $0 < \delta < \xi_1$, where $\xi_1 > 0$ is given as in Lemma 1.5.4. Also, let $T > 0$ and (τ_n) that we fixed in Assumptions (A). By the uniform inner convergence stated in Lemma 1.5.2 we know there exists $n_{\epsilon,\delta} > 0$ such that

$$|\phi(x, t+\tau_n)| < \epsilon \quad \text{for } x \in \Omega_{I,\delta},\ t \in [0,T],\ n \geq n_{\epsilon,\delta}. \tag{1.106}$$

Once we choose $\delta > 0$ we will obtain n_ϵ as above.

A first condition on the parameters will be that

$$T_1(\epsilon, \delta) = \frac{C - BC_1^m \delta - \epsilon}{A} < T. \tag{1.107}$$

Now, we consider the barrier function Φ and prove that $\phi(x, t+\tau_n) \leq \Phi(x,t)$, for a fixed $n \geq n_\epsilon$, on the set $\Omega_\delta \times (0, T_1)$, where $T_1 = T_1(\epsilon, \delta)$. More exactly, inequality (1.104) follows as a consequence of the parabolic maximum principle on this set.

Therefore, we have to check that this comparison is satisfied on the parabolic boundary formed by three pieces $\Omega_\delta \times 0 \cup \partial\Omega_{I,\delta} \times (0, T_1) \cup \partial\Omega \times (0, T_1)$.

1. *Comparison of ϕ with Φ at the initial section $t = 0$.* We want to obtain that

$$\phi(x, \tau_n) \leq \Phi(x, 0) = C - BV(x) \tag{1.108}$$

for all $x \in \Omega_\delta$. This is possible because of the uniform boundedness of ϕ

$$\left(\frac{C_0}{C_1}\right)^m - 1 = C_{2,m} \leq \phi \leq C_{3,m} = \left(\frac{C_1}{C_0}\right)^m - 1,$$

for all $x \in \Omega$ as a consequence of bounds (1.80). Now, we simply choose C sufficiently large and A, B to satisfy (1.99).

2. *Comparison of ϕ with Φ on the inner parabolic boundary.* This part of the boundary is given by the set

$$\partial\Omega_{I,\delta} \times (0, T_1) = \{(x,t) : \; x \in \Omega, \; d(x) = \delta, \, t \in (0, T_1)\}.$$

For (x,t) as before, $\Phi(x,t)$ is bounded as follows

$$C - At - BC_1^m \delta \le \Phi(x,t) \le C - At - BC_0^m \delta.$$

Let us fix $\epsilon > 0$ and $0 < \delta < \xi_1$ where $\xi_1 > 0$ is given as in Lemma (1.5.4). By (1.106)

$$\phi(x,t) < \epsilon \quad \text{for } x \in \Omega_{I,\delta}, \; t \in [0, T_1].$$

Thus one can obtain $\phi \le \Phi$ if

$$\epsilon \le C - At - BC_1^m \delta, \quad \forall t \in [0, T_1]. \tag{1.109}$$

Since C can not be small, this implies a choice for A and B compatible with (1.99) from the construction of the barrier. This can be realized by choosing $\delta > 0$ small. Once C and B are chosen it is sufficient to take At small.

3. *Comparison of ϕ with Φ on the outer lateral boundary.* This part of the boundary is given by the set $\partial\Omega \times [0, T_1]$, where we only know that $\phi = v^m/S^m - 1$ is bounded. As in [26] we can use an approximation trick using the solutions u_k of problems posed in the domain $\Omega^k \subset \Omega$. We will prove the desired comparison (1.104) for the function u_k and obtain it for u by passing to the limit.

We know that $u_k \nearrow u$ as $k \to \infty$ uniformly on the compact set $\overline{\Omega} \times [0, t]$, for every $t \le T_1$. Then

$$\phi_k = \frac{u_k^m}{\mathcal{U}^m} - 1 = -1 < 0 \quad \text{on } \partial\Omega^k \times [0, T_1],$$

where $\mathcal{U} = e^{-\lambda_1 t} S(x)$ is a separate variables solutions of the (DNLE) in Ω. Thus by (1.109) we have

$$\phi_k < 0 < C - At = \Phi \quad \text{on } \partial\Omega^k \times [0, T_1].$$

Steps 1 and 2 hold also for u_k since $u_k \le u$. Thus, by the parabolic comparison principle we obtain that $\phi_k \le \Phi$ on the region $\Omega^k \cap \Omega_\delta$ for $t \in [0, T_1]$. Passing to the limit when $k \to \infty$, we obtain $\phi \le \Phi$ on $\Omega_\delta \times [0, T_1]$.

We obtain in this way an improvement of the upper bound of ϕ near the boundary after some time delay given by

$$t \le T_1 = T_1(\epsilon, \delta) = \frac{C - BC_1^m \delta - \epsilon}{A},$$

which is the maximum that (1.109) allows. Notice that the delay time $T_1(\epsilon, \delta)$ does not depend on the time τ_n we fixed at the beginning.

Therefore, in order to choose the desired parameters we perform the following steps: we choose C sufficiently big to have (1.108). Then choose A and B to satisfy (1.99). Finally we choose δ small such that (1.109) and (1.107) hold , that is $t \le T_1(\epsilon, \delta) \le T$. $\square$

Better estimate from above for ϕ up to the boundary

Under the assumptions of Lemma 1.5.2 and Lemma 1.5.5 we deduce that for $t = \tau_n + T_1(\epsilon, \delta)$, where $T_1(\epsilon, \delta)$ is given by (1.105) and $n \ge n_\epsilon$, the REF ϕ satisfies the upper bound

$$\phi(x,t) \le \begin{cases} \epsilon, & d(x) > \delta; \\ \epsilon + BC_1^m \delta, & d(x) < \delta. \end{cases} \tag{1.110}$$

Therefore, by fixing $\epsilon > 0$, finding a barrier with constants A, B and C and then taking $\delta < \epsilon/(BC_1^m)$, we obtain the time $T_1(\epsilon, \delta)$ and the level n_ϵ such that for all $n \ge n_\epsilon$ we have

$$\phi(x, \tau_n + T_1) \le 2\epsilon \quad \forall x \in \Omega. \tag{1.111}$$

This means that $v(T_1 + \tau_n) \le (1+\epsilon)S$. The maximum principle implies now that the comparison is valid for all times $t \ge T_1 + \tau_{n_\epsilon}$. This proves that $\overline{c}_\infty \le c_*$, thus they are the same. One of the consequences is that c_* does not depend on the subsequence, therefore the whole family $v(t, \cdot)$ converges to $S = c_* f$ as $t \to \infty$. Moreover, we conclude the uniqueness of the profile $c_* f$ as well as the upper approximation stated in Proposition 1.5.1.

1.5.5 Construction of lower barriers

It remains to prove a similar bound for the REF ϕ from below. To this aim we define

$$\psi := -\phi = 1 - \frac{v^m(x,t)}{S^m(x)}.$$

We perform a similar approach as in the upper barrier case.

• *The parabolic equation of ψ*

$$-(p-1)(1-\psi)^{p-2}\psi_t = V^{-(p-1)}\Delta_p((1-\psi)V) + \lambda_1(1-\psi)^{p-1} \tag{1.112}$$

also rewritten as

$$\psi_t = -\frac{1}{(p-1)(1-\psi)^{p-2}} \left[V^{-(p-1)}\Delta_p((1-\psi)V) + \lambda_1(1-\psi)^{p-1} \right] \tag{1.113}$$

and equivalently

$$(p-1)(1-\psi)^{p-2}(1-\psi)_t = V^{-(p-1)}\Delta_p((1-\psi)V) + \lambda_1(1-\psi)^{p-1}. \tag{1.114}$$

By super-solution of equation (1.113) we understand a smooth function $\Psi : \Omega \times [0,\infty) \to \mathbb{R}^N$ such that

$$\Psi_t \geq -\frac{1}{(p-1)(1-\Psi)^{p-2}}\left[V^{-(p-1)}\Delta_p((1-\Psi)V) + \lambda_1(1-\Psi)^{p-1}\right].$$

This is equivalent to say that $1-\Psi$ is a classical subsolution for the equation (1.114).

• The function ψ is uniformly bounded in (x,t) for $t \geq 0$. This can be deduced from the estimates (1.80) on v and S, which is a stationary solution:

$$1-\left(\frac{C_1}{C_0}\right)^m = C_{2,m} \leq \psi \leq C_{3,m} = 1-\left(\frac{C_0}{C_1}\right)^m.$$

• In any interior region $\Omega_{I,\delta} \subset \Omega$, the function ψ satisfies

$$1-\psi = \frac{v^m}{V} > 0 \quad \text{in } \Omega_{I,\delta} \quad \text{for any } t \geq 0.$$

We use the same type of barrier as in the upper estimate case. However, differences appear.

Lemma 1.5.6. ***(Lower barrier)*** *We can choose positive constants A', B', C' so that for every $t_0 > 0$ the function*

$$\Psi(x,t) = C' - B'V(x) - A'(t-t_0), \tag{1.115}$$

is a super-solution to equation (1.112) *on a parabolic region near the boundary*

$$\Sigma_\Psi = \Sigma_{\Psi,\frac{1}{2}} = \left\{(x,t) \in \Omega_{\xi_0} \times (t_0,T_0) : 0 \leq \Psi(x,t) \leq \frac{1}{2}\right\},$$

and moreover $\Sigma_\Psi \subset \Omega_{\xi_2} \times (t_0,T_0)$, where $\xi_2 \leq \xi_0$.

Proof. We will prove that the function Ψ given by (1.115) is a supersolution for equation (1.112) on the parabolic region Σ_Ψ if we can find constants A', B' and C' such that

$$-(p-1)(1-\Psi)^{p-2}\Psi_t \leq V^{-(p-1)}\Delta_p((1-\Psi)V) + \lambda_1(1-\Psi)^{p-1}. \tag{1.116}$$

The first condition on the barrier function Ψ will be

$$0 \leq \Psi(x,t) \leq \frac{1}{2}, \quad (x,t) \in \Omega_{\xi_2} \times (t_0,T_0),$$

therefore it is sufficient to take

$$0 \leq C' - B'C_1^m\xi_0 - A'(t-t_0) \leq \Psi(x,t) \leq C' - A'(t-t_0) < \frac{1}{2}.$$

Moreover, this implies that

$$C' - A'(t-t_0) < 1.$$

From the beginning we search for a distance $\xi_2 \le \xi_0$ and we assume that $(x,t) \in \Sigma_\Psi \subset \Omega_{\xi_2} \times (t_0, T_0) \subset \Omega_{\xi_0} \times (t_0, T_0)$ where T_0 is such that

$$\frac{C' - 1/2}{A'} \le T_0 - t_0 \le \frac{C' - B' C_1^m \xi_0}{A'}, \tag{1.117}$$

and C' such that

$$C' \ge B' C_1^m \xi_0 \text{ and } C' \ge \frac{1}{2}.$$

The left hand side term of (1.116) is positive on Σ_Ψ,

$$-(p-1)(1-\Psi)^{p-2}\Psi_t = (p-1)A'(1-\Psi)^{p-2}. \tag{1.118}$$

The right hand side term (1.116) is of the form

$$V^{-(p-1)}\Delta_p((1-\Psi)V) + \lambda_1(1-\Psi)^{p-1} = V^{-(p-1)}\Delta_p g(V) + \lambda_1(1-\Psi)^{p-1},$$

where $g(z) = (1 - C' + B'z + A'(t-t_0))z$ and

$$\Delta_p g(V) = |g'(V)|^{p-2} g'(V)\Delta_p V + (p-1)|g'(V)|^{p-2} g''(V)|\nabla V|^p.$$

Properties of the function g

1. Function g is a convex parabola with zero values at the points $z = z_0$ and $z = 0$ and with the minimum value at the point $\frac{z_0}{2}$ where

$$z_0 := \frac{C' - 1 - A'(t-t_0)}{B'} < 0.$$

2. The derivatives are

$$g'(z) = 1 - C' + 2B'z + A'(t-t_0), \quad g''(z) = 2B'.$$

3. When applied to V, $g'(V(x))$ is positive and bounded sufficiently close the boundary:

$$0 < g'(0) = 1 - C' + A'(t-t_0) < g'(V(x)) = 1 - \Psi + B'V(x) < 1 + B'C_1^m\xi_0,$$

since $x \in \Omega_{\xi_2} \subset \Omega_{\xi_0}$ and $\Psi \ge 0$ in this domain.

Sufficient conditions for the parameters

Since V is a solution of the stationary problem (1.92) then $-\Delta_p V = \lambda_1 V^{p-1}$ and therefore the supersolution inequality (1.116) can be rewritten as

$$\begin{aligned}(p-1)A'(1-\Psi)^{p-2} \le\\ -\lambda_1|g'(V)|^{p-2}g'(V) + 2B'(p-1)V^{-(p-1)}|g'(V)|^{p-2}|\nabla V|^p + \lambda_1(1-\Psi)^{p-1}.\end{aligned} \tag{1.119}$$

Next, the idea is that when we are sufficiently close to the boundary, $V^{-(p-1)}$ will be large enough and then the right hand side term of (1.119) will be positive and large enough. More exactly, on Σ_Ψ

$$2B'(p-1)V^{-(p-1)}|g'(V)|^{p-2}|\nabla V|^p \geq 2B'(p-1)K_1^p(g'(V))^{p-2}(C_1^m\xi_2)^{-(p-1)}.$$

Since $g'(V(x)) = 1-\Psi+B'V(x) \leq 1+B'C_1^m\xi_0$, then it is sufficient to find A', B', C', ξ_2 such that

$$(p-1)A'(1-\Psi)^{p-2} + \lambda_1(1+B'C_1^m\xi_0)^{p-1} \leq 2B'(p-1)K_1^pC_1^{-1}(g'(V))^{p-2}\xi_2^{-(p-1)}.$$

- **Case** $p \geq 2$

In this case $(g'(V))^{p-2} \geq (g'(0))^{p-2} \geq (1-C'+A'(t-t_0))^{p-2}$. Therefore it is sufficient to take

$$(p-1)A' + \lambda_1(1+B'C_1^m\xi_0)^{p-1} \leq 2B'(p-1)K_1^pC_1^{-1}(1-C'+A'(t-t_0))^{p-2}\xi_2^{-(p-1)}$$

that is ξ_2 is sufficiently small

$$\xi_2 \leq \left[\frac{2B'(p-1)K_1^pC_1^{-1}(1-C'+A'(t-t_0))^{p-2}}{(p-1)A'+\lambda_1(1+B'C_1^m\xi_0)^{p-1}}\right]^{1/(p-1)}. \tag{1.120}$$

- **Case** $1 \leq p \leq 2$

In this case $(g'(V))^{p-2} \geq (1+B'V(x))^{p-2} \geq (1+B'C_1^m\xi_0)^{p-2}$ therefore it is sufficient to take

$$\begin{aligned}(p-1)A'(1-C'+A(t-t_0))^{p-2} + \lambda_1(1+B'C_1^m\xi_0)^{p-1}\\ \leq 2B'(p-1)K_1^pC_1^{-1}\left(1+B'C_1^m\xi_0\right)^{p-2}\xi_2^{-(p-1)}\end{aligned}$$

that is ξ_2 is sufficiently small

$$\xi_2 \leq \left[\frac{2B'(p-1)K_1^pC_1^{-1}\,(1+B'C_1^m\xi_0)^{p-2}}{(p-1)A'(1-C'+A'(t-t_0))^{p-2}+\lambda_1(1+B'C_1^m\xi_0)^{p-1}}\right]^{1/(p-1)}. \tag{1.121}$$

Summary: we choose $C' \geq \frac{1}{2}$, then $B' > 0$ such that $C' \geq B'C_1^m\xi_0$, A' arbitrary and T_0 defined by (1.117). Finally we take ξ_2 the minimum satisfying $\xi_2 \leq \xi_0$ and the upper bounds (1.120) and (1.121).

□

Lemma 1.5.7. *Let Ψ be the barrier function introduced in Lemma 1.5.6, given by*

$$\Psi(x,t) = C' - B'V(x) - A't.$$

Let $T > 0$ and $\tau_n \to \infty$ be a sequence along which the REF converges to 1 as stated in Assumptions (A). Then for any $\epsilon > 0$ we can choose $n_\epsilon > 0$ and positive constants A',

B', C' and δ as in Lemma 1.5.6 such that

$$\psi(x, t+\tau_n) \leq \Psi(x,t), \quad \forall x \in \Omega_\delta,\ \forall n \geq n_\epsilon,\ \forall t \in [0, T_2], \tag{1.122}$$

where

$$T_2 = T_2(\epsilon, \delta) = \frac{C' - B'C_1^m\delta - \epsilon}{A'}. \tag{1.123}$$

Proof. Let $T > 0$ and (τ_n) that we fixed in Assumptions (A). We fix $\epsilon > 0$ and consider $0 < \delta < \xi_2$ where $\xi_2 > 0$ is given as in Lemma 1.5.6.

We adapt the proof of Lemma 1.5.4 by writing the estimates in terms of the function $\psi = -\phi$. By the uniform inner convergence stated in Lemma 1.5.2 we know there exists $n_\epsilon > 0$ such that

$$|\psi(x, \tau_n + t)| < \epsilon \quad \text{for } x \in \Omega_{I,\delta},\ t \in [0, T],\ n \geq n_\epsilon. \tag{1.124}$$

We impose a first condition on the parameters A', B', C'

$$T_2(\epsilon, \delta) = \frac{C' - B'C_1^m\delta - \epsilon}{A'} < T. \tag{1.125}$$

This new time $T_2(\epsilon, \delta)$ must satisfy the inequality (1.117) from Lemma 1.5.6 which reduces to

$$B'C_1^m\delta + \epsilon \leq 1/2 \quad \text{and} \quad B'C_1^m(\xi_0 - \delta) \leq \epsilon.$$

Since $\delta < \xi_2 \leq \xi_0$, it is sufficient to take

$$B'C_1^m\xi_0 + \epsilon \leq 1/2 \quad \text{and} \quad B'C_1^m(\xi_0 - \delta) \leq \epsilon. \tag{1.126}$$

Now, we consider the barrier function Ψ constructed in Lemma 1.5.6 and prove that $\psi(x, t+\tau_n) \leq \Psi(x,t)$, for a fixed $n \geq n_\epsilon$, on the set $\Omega_\delta \times (0, T_2)$, where $T_2 = T_{2,\epsilon,\delta}$. More exactly, inequality (1.122) follows as a consequence of the parabolic maximum principle on this set.

Therefore, we have to check that this comparison is satisfied on the parabolic boundary formed by three pieces $\Omega_\delta \times 0 \cup \partial\Omega_{I,\delta} \times (0, T_2) \cup \partial\Omega \times (0, T_2)$.

1. *Comparison of ψ with Ψ at the initial section $t = 0$.* We want to obtain that

$$\psi(x, \tau_n) \leq \Psi(x, 0) = C' - B'V(x) \quad \text{for all}\, x \in \Omega_\delta.$$

This is possible because of the uniform boundedness of ψ,

$$1 - \left(\frac{C_1}{C_0}\right)^m = C_{3,m} \leq \psi \leq C_{4,m} = 1 - \left(\frac{C_0}{C_1}\right)^m \quad \text{for all}\, x \in \Omega$$

as a consequence of bounds (1.80). Now, we simply take B' arbitrary and $C' > 1 + B'C_1^m\xi_0$.

2. *Comparison of ψ with Ψ on the inner parabolic boundary.* This part of the boundary is given by the set

$$\partial\Omega_{I,\delta} \times (0, T_2) = \{(x,t) : \ x \in \Omega, \ d(x) = \delta, \, t \in (0, T_2)\}.$$

For (x,t) as before, $\Psi(x,t)$ is bounded as follows

$$C' - A't - B'C_1^m \delta \le \Psi(x,t) \le C' - A't - B'C_0^m \delta.$$

Let us fix $\epsilon > 0$ and $0 < \delta < \xi_2$ where $\xi_2 > 0$ is given as in Lemma (1.5.6). By (1.124)

$$\psi(x, t + \tau_n) < \epsilon \quad \text{for } x \in \Omega_{I,\delta}, \ t \in [0, T_2].$$

Thus one can obtain $\psi \le \Psi$ if

$$\epsilon \le C' - A't - B'C_1^m \delta, \quad \forall t \in [0, T_2], \tag{1.127}$$

or equivalently

$$t \le \frac{C' - B'C_1^m \delta - \epsilon}{A'} = T_2(\epsilon, \delta).$$

3. *Comparison of ψ with Ψ on the outer lateral boundary.* This part of the boundary is given by the set $\partial\Omega \times [0, T_2]$, where we only know that $\psi = 1 - v^m/S^m$ is bounded. Here we will use an approximation trick using the solutions u_k of problems posed in an extended domain $\Omega^k \supset \Omega$. As in Lemma 5.5 we prove the desired comparison (1.122) for the approximating function u_k. We know that $u^k \searrow u$ as $k \to \infty$ uniformly on the compact set $\overline{\Omega} \times [0, t]$, for every $t \le T_2$. Formally

$$\psi_k = 1 - \frac{u_k^m}{\mathcal{U}^m} = -\infty \quad \text{on } \partial\Omega \times [0, T_2],$$

where $\mathcal{U} = e^{-\lambda_1 t} S(x)$ is a separate variables solutions of the (DNLE) in Ω. On the other hand, on $\partial\Omega \times [0, T_2]$,

$$\Psi(x,t) = C' - A't \ge C' - A'T_2 = \epsilon + B'C_1^m \delta$$

and thus the comparison $\psi_k \le \Psi$ holds true on the outer lateral boundary. Finally, steps 1 and 2 hold also for u_k since $u_k \ge u$ and thus, by parabolic comparison we obtain that $\phi_k \le \Phi$ on $\Omega_\delta \times [0, T_2]$. Passing to the limit when $k \to \infty$ we obtain $\psi \le \Psi$ on $\Omega_\delta \times [0, T_2]$.

We obtain in this way an improvement of the lower bound of ϕ near the boundary after some time delay given by

$$t \le T_2(\epsilon, \delta) = \frac{C' - B'C_1^m \delta - \epsilon}{A'},$$

which is the maximum that (1.127) allows. Notice that the delay time $T_2(\epsilon,\delta)$ does not depend on the time sequence (τ_n) we fixed at the beginning.

Therefore, in this case of comparison with lower barriers, in order to choose the desired parameters satisfying both conditions of Lemma 1.5.6 and the conditions above we perform the steps: we choose $C' > 1$, then choose B' the minimum such that $1 + B'C_1^m\xi_0 < C'$ and $B'C_1^m\xi_0 + \epsilon \le 1/2$. Then we define $\delta < \xi_2$ such that $B'C_1^m(\xi_0 - \delta) \le \epsilon$ and we choose A' such that condition (1.125) holds, i.e. $C' - B'C_1^m\delta - \epsilon < A'T$.

□

Better estimate from below for ϕ up to the boundary

Under the assumptions of Lemma 1.5.2 and Lemma 1.5.7 we deduce that for $t = \tau_n + T_2(\epsilon,\delta)$, for a fixed $n \ge n_\epsilon$, where $T_2(\epsilon,\delta)$ is given by (1.123), the REF $\psi = -\phi$ satisfies the upper bound

$$\psi(x,t) \le \begin{cases} \epsilon, & d(x) > \delta; \\ \epsilon + B'C_1^m\delta, & d(x) < \delta. \end{cases} \tag{1.128}$$

Therefore, by fixing $\epsilon > 0$, finding a barrier Ψ with constants A', B' and C' and then taking $\delta < \epsilon/(B'C_0^m)$, we obtain a time $T_2(\epsilon,\delta)$ and a n_ϵ such that for all $n \ge n_\epsilon$ we have

$$\psi(x, \tau_n + T_2) \le \epsilon \quad \forall x \in \Omega. \tag{1.129}$$

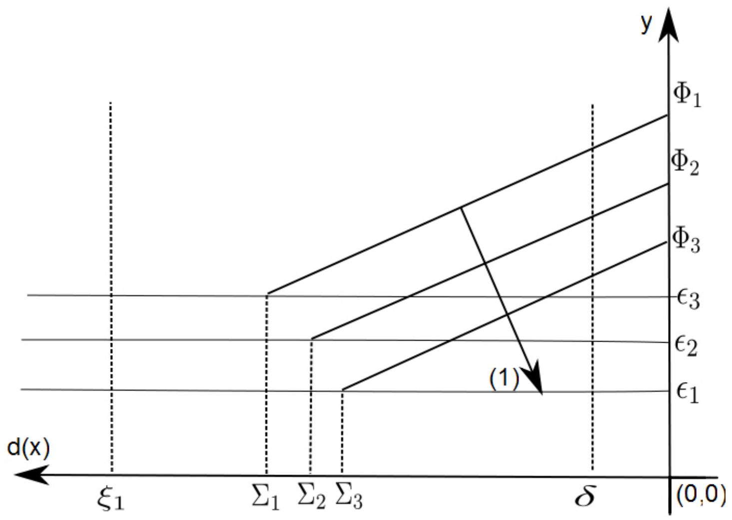


FIGURE 1.4: Idea of the behaviour of the barriers: y-axis: values of $\Phi(x,t)$, x-axis: values of $d(x) = d(x,\partial\Omega)$, i.e. the distance to the boundary. Σ_i: the points where the barrier satisfies $\Phi(x,t) = \epsilon_i$. Notations: ϵ_i: different values of ϵ(decreasing with i=1,2,3) give different barriers Φ_i decreasing with ϵ as the arrow (1) indicates. ξ_1 and δ as in Lemma 1.5.4.

1.6 Appendix A

1.6.1 Two convergence results

The following lemma can be easily proved with basic computations.

Lemma 1.6.1. ***(Property of the convergence in measure)*** *Let $f \subset L^p(\Omega)$ and $(f_n)_n \subset L^p(\Omega)$ a sequence of functions such that*

- $f_n \to f$ *in measure;*
- $\|f_n\|_{L^p(\Omega)}$ *uniformly bounded.*

Then

$$f_n \to f \text{ in } L^q(\Omega), \quad \text{for every } 1 \leq q < p.$$

Another useful result in our proofs is a lemma concerning nonlinear monotone operators due to Brezis [34].

Lemma 1.6.2. *Let A be a maximal monotone operator on a Hilbert space H. Let Z_n and W_n be measurable functions from Ω (a finite measure space) into H. Assume $Z_n \to Z$ a.e. on Ω and $W_n \rightharpoonup W$ weakly in $L^1(\Omega; H)$. If $W_n(x) \in A(Z_n(x))$ a.e. on Ω, then $W(x) \in A(Z(x))$ a.e. on Ω.*

1.6.2 Regularity

Concerning the regularity of the solution u of the (DNLE-d), we refer for example to [72], [89], [112].

(Theorem 2.1 from [72]- inner Hölder estimate) Let u be a weak solution of the (DNLE-d). Then

$$u \in C^{\alpha/p,\alpha}_{loc}(\Omega \times [0,T]) \text{ for some } \alpha \in (0,1). \tag{1.130}$$

Moreover, for every cylinder $Q' = \Omega' \times [\epsilon, T]$, $\overline{\Omega'} \subset \Omega$, $\epsilon > 0$, we have

$$\sup_{(x,t)(t',x') \in Q'} \frac{|u(x,t) - u(x',t')|}{(|t-t'|^m + |x-x'|^m)^{\alpha/p}} \leq K, \tag{1.131}$$

where $\alpha \in (0,1)$ and $K > 0$ depend only on T, Ω' and data.

(Theorem 2.2. from [72]- Hölder estimate up to the boundary) If Ω has regular boundary then

$$u \in C^{\alpha/p,\alpha}_{loc}(\Omega \times [0,T]) \text{ for some } \alpha \in (0,1)$$

and u satisfies an estimate similar to (1.131).

1.6.3 Distance to the boundary function

We collect some properties of the *distance to the boundary function* for which we refer to [63] and [92]. Let $d : \overline{\Omega} \to [0, +\infty)$ be given by

$$d(x) = \mathrm{dist}(x, \partial\Omega) = \min\{|x - z| : z \in \partial\Omega\},$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^d$. In terms of $d(x)$ we define the sets

$$\Omega_{I,r} = \{x \in \overline{\Omega} : d(x) > r\},$$

$$\Omega_r = \Omega \setminus \overline{\Omega_{I,r}} = \{x \in \overline{\Omega} : d(x) < r\},$$

and we remark that, for all small $r > 0$,

$$\partial\Omega_{I,r} = \{x \in \Omega : d(x) = r\}.$$

Lemma 1.6.3. ***(Properties of the distance to the boundary)*** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ of class C^2. Then*

1. *There is a constant $\xi_0 \in \mathbb{R}_+$ such that for every $x \in \Omega_{\xi_0}$, there is a unique $h(x) \in \partial\Omega$ which realizes the distance*

$$d(x) = |x - h(x)|.$$

Moreover, $d(x) \in C^2(\Omega_{\xi_0})$, and for all $r \in [0, \xi_0)$ the function $H_r : \partial(\overline{\Omega_r}) \cap \Omega \to \partial\Omega$ defined by $H_r(x) = h(x)$ is a homeomorphism.

2. *Function $d(x)$ is Lipschitz with constant 1, i.e. $|d(x) - d(y)| \leq |x - y|$. Moreover,*

$$0 < c \leq |\nabla d(x)| \leq 1, \quad \forall x \in \Omega_{\xi_0}, \tag{1.132}$$

and there exists a constant $K > 0$ such that

$$-K \leq \partial^2_{ij} d(x) \leq K, \quad \forall x \in \Omega_{\xi_0}, \forall i, j = 1, N.$$

Notice that this ξ_0 can be characterized as follows:

$$\xi_0 = \{\min_{\overline{x} \in \partial\Omega} \max_{r > 0} r : B_r(\overline{x} + r\nu) \text{ is tangent at } \partial\Omega \text{ in } \overline{x}\},$$

where ν is the inward unit normal at $\partial\Omega$ in x_0. We observe that $d(x_0 + r\nu) = R$ and

$$\Omega_r \subset \bigcup_{y \in \partial\Omega_{I,r}} B_r(y). \tag{1.133}$$

Throughout the paper we have constantly used the notation ξ_0 with the properties stated above.

1.7 Comments and open problems

• In this work we have discussed only the slow diffusion case $m(p-1) > 1$ and the quasilinear case $m(p-1) = 1$. The fast diffusion case $m(p-1) < 1$ produces different results and thus it needs different techniques. For this last case we mention the results of Savaré and Vespri ([90]) about the asymptotic behaviour of the (DNLE) in the singular case. In that paper the authors prove the convergence to an asymptotic profile for a sequence of times $t_n \to T$, T being the extinction time. The uniform convergence for all times and the rate of convergence in the fast diffusion case, for both (DNLE) and (PLE), remain an open problem at this moment. However, the fast diffusion regimes for the PME have been much discussed in the literature, we mention the results of Bonforte, Dolbeaut, Grillo, Vázquez [23, 25, 26, 32]; Feiresl and Simondon [58] on the PME. Partial results on the PLE were given in [31].

• We presented only a formal description of the self similar solutions of the (DNLE) in the case $m(p-1) > 1$. We do not offer a complete characterization of such functions since this is beyond the purpose of our paper. The problem is interesting and it deserves a separate study.

• Our result in the quasilinear case is not as sharp as the result in the degenerate case. Indeed, we only prove convergence in relative error. The problem of a rate of convergence similar to Theorem 1.1.1 is still open, except in the linear case $m = 1$, $p = 2$, where a representation as infinite series follows from the Fourier analysis of the solution.

• For the Cauchy problem $u_t = \Delta_p u^m$, with the spatial domain $\mathbb{R}^N$, $N \geq 3$, and integrable initial data, the asymptotic behavior is given by a Barenblatt-type solution in the range of parameters $1 < m(p-1) + (p/N)$ when the total mass of the solution is conserved. This has been done by Agueh, Blanchet and Carrillo in [2], where they prove the L^1-algebraic decay of the non-negative solution to a Barenblatt-type solution for the case $1 < m(p-1) + (p/N) < 1 + 1/N$ and they estimate the rate of convergence. In [3–5], Agueh proves convergence with rates in the range $m(p-1) + (p/N) > 1 + 1/N$.

In [51] Del Pino and Dolbeault prove the asymptotic behaviour (convergence with rates) of solution to the Cauchy problem in the case $N \geq 2$, $1 < p < N$, $1 + \frac{1}{N} \leq m(p-1) + \frac{p}{N} \leq p(1 + \frac{1}{N})$. Their proof uses entropy estimates based on a sub-family of the Gagliardo–Nirenberg inequalities.

As for the remaining case $m(p-1) + (p/N) < 1$, the problem has extinction in finite time and there are not many references in the literature for (DNLE).

In the case $p = 2$, when we recover the (PME), we mention the papers: [41] by Carrillo and Toscani concerning the asymptotic behaviour of the Cauchy problem when $m > 1$ (PME), and [42] for the case $m < 1$, that is the Fast Diffusion Equation.

• More general problems of this type can be considered by similar techniques. Let us mention the doubly nonlinear equation with mixed boundary conditions or p-Laplacian type equations with variable coefficients.

Chapter 2

Nonlocal diffusion. The Fractional Porous Medium Equation

The model of nonlocal and nonlinear diffusion that we consider in this thesis is the *Fractional Porous Medium Equation* $u_t + (-\Delta)^s u^m = 0$. For a description on recent progress on this subject we refer to the survey papers by Vázquez [108, 109].

We consider the initial data problem

$$\begin{cases} u_t + (-\Delta)^s(|u|^{m-1}u) = 0 & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

with data $u_0 \in L^1(\mathbb{R}^N)$ and exponents $0 < s < 1$ and $m > 0$. In [47] existence and uniqueness of a weak solution is established for $m > m_c = (N-2s)_+/N$ giving rise to an L^1-contraction semigroup which depends continuously on the exponent of fractional differentiation and the exponent of the nonlinearity. Recently in [48], the authors proved the classical regularity.

Contrary to usual porous medium flows, the fractional version has infinite speed of propagation for all exponents $0 < s < 1$ and $m > 0$. Positivity of the solution for any $m > 0$ corresponding to non-negative data has been proved in [47] and quantitative positivity estimates in [28].

In this thesis, I will consider the evolution problem for non-negative solutions which will be referred by the name (FPME):

$$\begin{cases} u_t(x,t) + (-\Delta)^s u^m(x,t) = 0 & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{FPME}$$

2.1 Barenblatt solutions of the Fractional Porous Medium Equation

An important tool describing the properties of the (FPME) is the study of the fundamental solutions, also called Barenblatt solutions. In [107], Vázquez proves existence, uniqueness and main properties of such fundamental solutions of the equation

$$u_t + (-\Delta)^s u^m = 0, \tag{2.2}$$

taking as initial data a multiple of the Dirac delta

$$u(x,0) = M\delta(x),$$

where $M > 0$ is the mass of the solution. We will give here a short description of these functions and recall their main properties.

Next, we recall Theorem 1.1 from [107].

Theorem 2.1.1. *For every choice of parameters $s \in (0,1)$ and $m > m_c = \max\{(N-2s)/N, 0\}$, and every $M > 0$, Equation (2.2) admits a unique fundamental solution with initial condition $M\delta(x)$; it is a nonnegative and continuous weak solution for $t > 0$ and takes the initial data in the sense of Radon measures. Such solution has the self-similar form*

$$B_M(x,t) = t^{-\alpha} F_M(|x| t^{-\beta}) \tag{2.3}$$

for suitable α and β that can be calculated in terms of N and s in a dimensional way, precisely

$$\alpha = \frac{N}{N(m-1)+2s}, \quad \beta = \frac{1}{N(m-1)+2s}. \tag{2.4}$$

The profile function $F_M(r)$, $r \geq 0$, is a bounded and Hölder continuous function, it is positive everywhere, it is monotone and goes to zero at infinity.

In what follows we denote by F_M the profile corresponding to the Barenblatt solution with mass M, as stated in the above theorem. By Theorem 2.1.1 there exists a unique self-similar solution $B_1(x,t)$ with mass $M = 1$ of Problem (2.2) and moreover, it has the form $B_1(x,t) = t^{-\alpha} F_1(|x| t^{-\beta})$. Let $B_M(x,t)$ the unique self-similar solution of Problem (2.2) with mass M. Such function will be of the form

$$B_M(x,t) = M B_1\left(x, M^{m-1} t\right),$$

which can be written in terms of the profile F_1 as

$$B_M(x,t) = M^{1-(m-1)\alpha} t^{-\alpha} F_1\left(\left(M_1^{m-1} t\right)^{-\beta} |x|\right).$$

Moreover, the precise characterization of the profile F_M is given by Theorem 8.1 of [107].

Theorem 2.1.2. *For every $m > m_1 = N/(N+2s)$ we have the asymptotic estimate*

$$\lim_{r\to\infty} F_M(r)r^{N+2s} = C_1 M^\sigma, \tag{2.5}$$

where $M = \int_{\mathbb{R}^N} F(x)dx$, $C_1 = C_1(m,N,s) > 0$ and $\sigma = (m-m_1)(N+2s)\beta$. On the other hand, for $m_c < m < m_1$, there is a constant $C_\infty(m,N,s)$ such that

$$\lim_{r\to\infty} F_M(r)r^{2s/(1-m)} = C_\infty. \tag{2.6}$$

The case $m = m_1$ has a logarithmic correction. The profile F_M has the upper bound

$$F_M(r) \le Cr^{-N-2s+\epsilon}, \quad \forall r > 0 \tag{2.7}$$

for every $\epsilon > 0$, and the lower bound

$$F_M(r) \ge Cr^{-N-2s}\log r, \quad \text{for all large } r. \tag{2.8}$$

We state now some properties of the profile $F_M(r)$, $r \ge 0$, obtained as consequences of formula (2.5), that we will use in what follows. Let us consider first the case $m > m_1$.

1. F_1 attains its maximum when $r = 0$ i.e. $F_M(r) \le F_M(0)$, for all $r \ge 0$.

2. There exists $K_1 > 0$ such that

$$F_M(r) \le K_1 r^{-(N+2s)}, \quad \forall r > 0. \tag{2.9}$$

3. There exists $K_2 > 0$ such that

$$F_M(r) \ge K_2(1 + r^{N+2s})^{-1}, \quad \forall r \ge 0. \tag{2.10}$$

Similar estimates hold also in the case $m_c < m < m_1$, and the corresponding tail behaviour is different, $F_M(r) \sim r^{-2s/(1-m)}$. This will have an effect in the different results we get for the generalized KPP problem.

As a consequence, the author also proves that the asymptotic behaviour of general solutions of Problem (2.1) is represented by such special solutions, as described in Theorem 10.1 from [107].

Theorem 2.1.3. *Let $u_0 = \mu \in \mathcal{M}_+(\mathbb{R}^N)$, $M = \mu(\mathbb{R}^N)$ and let u be the solution of* (2.1) *and B_M be the self-similar Barenblatt solution with mass M. Then we have*

$$\lim_{t\to\infty} t^\alpha |u(x,t) - B_M(x,t;M)| = 0$$

and the convergence is uniform in $\mathbb{R}^N$.

The limit m goes to infinity is the Mesa Problem, see Vázquez [102].

2.2 Lower estimates for nonnegative solutions in the case $m_c < m < 1$

The fact that solutions of the (FPME) with nonnegative initial data become immediately positive for all times $t > 0$ in the whole space has been proved in [46, 47]. Such result is true not only for $0 < s < 1$ and $m > 1$, but also for $0 < s < 1$ and $m > m_c = (N-2s)_+/N$, this lower restriction on m aimed at avoiding the possibility of extinction in finite time.

Precise quantitative estimates of positivity for $t > 0$ on bounded domains of $\mathbb{R}^N$ have been obtained in the recent paper [28]. The estimates of that reference are also precise in describing the behaviour as $|x| \to \infty$ when $m < 1$ (fast diffusion), but they are not relevant to establish the far-field behaviour for $m > 1$. We recall that in the limit $s \to 1$ with $m > 1$ fixed we get the standard porous medium equation, where positivity at infinity for all nonnegative solutions is false due to the property of finite propagation, cf. [103]. This explains that some special characteristic of fractional diffusion must play a role if positivity is true.

We recall the notations: $m_c = (N-2s)_+/N$, $\beta = 1/[2s + N(m-1)] > 0$ for $m > m_c$. The results we quote are valid for initial data in a weighted space $u_0 \in L^1(\mathbb{R}^N, \varphi dx)$, where φ satisfies the following conditions:

Assumption (A). The function $\varphi \in C^2(\mathbb{R}^N)$ is a positive real function that is radially symmetric and decreasing in $|x| \geq 1$. Moreover φ satisfies

$$0 \leq \varphi(x) \leq |x|^{-\alpha} \quad \text{for } |x| >> 1 \text{ and } N - \frac{2s}{1-m} < \alpha < N + \frac{2s}{1-m}.$$

We recall now Theorem 4.1 from [28] giving local lower bounds for the solution of the diffusion problem.

Theorem 2.2.1 (Local lower bounds). *Let $R_0 > 0$, $m_c < m < 1$ and let $0 \leq u_0 \in L^1(\mathbb{R}^N, \varphi dx)$, where φ is as in Assumption (A). Let $u(\cdot, t) \in L^1(\mathbb{R}^N, \varphi dx)$ be a very weak solution to the Cauchy Problem* (FPME)*, corresponding to the initial datum u_0. Then there exists a time*

$$t_* := C_* R_0^{\frac{1}{\beta}} \|u_0\|_{L^1(B_{R_0})}^{1-m} \tag{2.11}$$

such that

$$\inf_{x \in B_{R_0/2}} u(x,t) \geq K_1 R_0^{-\frac{2s}{1-m}} t^{\frac{1}{1-m}} \text{ if } 0 \leq t \leq t_*, \tag{2.12}$$

and

$$\inf_{x \in B_{R_0/2}} u(x,t) \geq K_1 \frac{\|u_0\|_{L^1(B_{R_0})}^{2s\beta}}{t^{N\beta}} \text{ if } t \geq t_*. \tag{2.13}$$

The positive constants C_*, K_1, K_2 depend only on m, s and $N \geq 1$.

The previous estimates, computed for $t = t_*$ rewrite as

$$\inf_{x \in B_{R_0/2}} u(x,t) \geq K_1 C_*^{\frac{1}{1-m}} \|u_0\|_{L^1(B_{R_0})} R_0^{-N}. \tag{2.14}$$

Then, if R_0 increases, the lower bound will decrease.

Concerning quantitative lower estimates for large $|x|$, we recall Theorem 4.3 from [28].

Theorem 2.2.2 (**Global Lower Bounds when** $m_1 < m < 1$)**.** *Under the conditions of Theorem 2.2.1 we have in the range* $m_1 < m < 1$

$$u(x,t) \geq \frac{C(t)}{|x|^{N+2s}} \ \textit{when } |x| >> 1, \tag{2.15}$$

valid for all $0 < t < T$ *with some bounded function* $C > 0$ *that depends on* t, T *and on the data.*

Theorem 2.2.3 (**Global Lower Bounds when** $m_c < m < m_1$)**.** *Under the conditions of Theorem 2.2.1 we have in the range* $m_c < m < m_1$

$$u(x,t_0) \geq C(t)|x|^{-2s/(1-m)} \tag{2.16}$$

if $|x| \geq R$ *and* $0 < t < t_0$.

The lower estimates for exponents $m > 1$ need a new analysis that we supply in the next section.

2.3 Lower parabolic estimate in the case $m > 1$

We consider the (FPME) equation for $x \in \mathbb{R}^N$ and $t > 0$ with nonnegative and integrable initial data

$$u(x,0) = u_0(x)\,, \tag{2.17}$$

and we also assume that u_0 is bounded and has compact support or decays rapidly as $|x| \to \infty$. We want to describe the behaviour of the solution $u(x,t) > 0$ as $|x| \to \infty$, more precisely its rate of decay, for small times $t > 0$. We take $m > 1$ since the study of positivity for $m \leq 1$ was dealt with in previous results.

The first step in our asymptotic positivity analysis of solutions of (FPME) is to ensure that solutions with positive data remain positive and they have a precise tail behaviour from below, which is based on a delicate subsolution construction.

Theorem 2.3.1. *Let* $m > 1$ *and let* $u(x,t)$ *be a solution to Equation* (FPME) *with initial data* $u_0(x) \geq 0$ *such that* $u_0(x) \geq 1$ *in the ball* $B_1(0)$. *Then there is a time* $t_1 > 0$ *and constants* $C_*, R > 0$ *such that*

$$u(x,t) \geq C_*\, t\, |x|^{-(N+2s)} \tag{2.18}$$

if $|x| \geq R$ *and* $0 < t < t_1$.

Proof. By comparison we may consider some smaller initial data u_0, such that $0 \leq u_0(x) \leq 1$ and $u_0(x) = 1$ in the ball of radius 2. Moreover, u_0 is smooth. By the results of [47] we know that $u(x,t) \in C^\alpha(\mathbb{R}^N \times [0,T])$ and $u(x,t) > 0$ for all $x \in \mathbb{R}^N$ and $t > 0$. We have that $u(x,t) \geq 1/2$ in the ball of radius $1/2$ for all small times $0 < t < t_0$.

• We want to construct a sub-solution of the form

$$U^m(x,t) = G(|x|) + t^m\, F^m(|x|).$$

We want to choose $G \geq 0$ and $F \geq 0$ in such a way that U will be a formal sub-solution of the (FPME) in a domain of the form $Q = \{|x| \geq 1/2, 0 < t < t_1\}$, i.e., we want $U_t + (-\Delta)^s U^m \leq 0$ in Q. Note that

$$U_t = (G(|x|) + t^m\, F^m(|x|))^{(1/m)-1} t^{m-1} F^m(|x|) \leq F(|x|).$$

We also have, with $L_s = (-\Delta)^s$,

$$L_s U^m = L_s G(|x|) + t^m L_s F^m(|x|).$$

We take F positive, smooth and $F(r) \sim r^{-(N+2s)}$ as $r \to \infty$ to get the desired conclusion after the comparison argument: $u(x,t) \geq U(x,t) \geq ct\, r^{-(N+2s)}$ if r is large and $t \sim 0$. For later use, let us say that $F \leq C_2 r^{-(N+2s)}$ for $r > 1/2$. Since $m > 1$ we can choose F smooth so that $L_s F^m = O(r^{-(N+2s)})$ for $r > 1/2$ (use the asymptotic estimates like the first lemma in [28])

We will take $G(r) = 0$ for $r = |x| \geq 1/2$ so that $U(x,t) = t\, F(|x|)$ there. If G is also smooth we have $L_s G$ bounded and $L_s G \sim -C_1 r^{-(N+2s)}$ as $r \to \infty$. In Lemma 4.7.1 we construct similar functions G and F. By contracting G in space, $\widetilde{G}(x) = G(kx)$, $k > 0$, we may then say that $L_s G \leq -C_1 r^{-(N+2s)}$ for $r > 1/2$. Then we will have for $r > 1/2$ and $0 < t << 1$ that

$$U_t + L_s U^m \leq F + L_s G + t^m L_s F^m \leq C_2 r^{-(N+2s)} - C_1 r^{-(N+2s)} + t^m L_s F^m \leq$$

$$(C_2 + \varepsilon) r^{-(N+2s)} - C_1 r^{-(N+2s)} \leq 0$$

if $C_1 > C_2$. We can choose G large such that C_1 is large enough.

• We now want to prove that $U(x,t) \leq u(x,t)$ in Q. This involves a Comparison Principle for the nonlocal equation (FPME), similar to ones in Lemma 3.4.1 and Lemma 4.6.2. More exactly, if the following conditions are satisfied: (i) $U(x,0) \leq u(x,0)$ in $\mathbb{R}^N$; (ii) $u_t + L_s u^m = 0$, $U_t + L_s U^m \leq 0$ in Q; (iii) $U(x,t) \leq u(x,t)$ in $\big(\mathbb{R}^N \times (0,t_1)\big) \setminus Q$, then we obtain that $U(x,t) \leq u(x,t)$ in Q.

Apart from the sub-solution condition that we have checked, we need suitable comparison at the parabolic boundary $\big(\mathbb{R}^N \times (0,t_1)\big) \setminus Q = \{|x| \leq 1/2, 0 < t < t_1\}$. For

$|x| = 1/2$ we have that

$$U(1/2, t) = tF(1/2) \le 1/2.$$

For $|x| < 1/2$, and $0 < t < t_1$ we have

$$U(x,t) \le \left(\sup_{|x|<1/2} G(|x|) + t^m \sup_{|x|<1/2} F^m(|x|) \right)^{1/m} \le 1/2 \le u(x,t)$$

if we ensure G and F are bounded from above by appropriate constants.

The proof of $U \le u$ in Q is based on a contradiction argument at the first point of contact between u and U. This can be done as in [28] (where it was applied to fast diffusion equations of fractional diffusion type) if the solution we have is a bit smooth: u_t and $L_s u^m$ must be continuous and the equation must be satisfied pointwise there. This regularity is true by [48]. A similar argument will be used later in the proof of Proposition 4.6.2.

Alternatively, we may use Implicit Time Discretization with a sequence of approximations. The justification of the method in the elliptic case is done in the paper of Vázquez and Volzone [111] on symmetrization techniques. □

Remark. The level $u_0(x) \ge 1$ in the ball $B_1(0)$ can be replaced by $u_0(x) \ge \varepsilon > 0$ in any other ball by means of translation and scaling. In this way the result is true for all continuous and nonnegative initial data u_0, of course nontrivial.

2.4 The linear diffusion problem $m = 1$

We will need a number of facts about the linear diffusion equation for $0 < s < 1$,

$$U_t + (-\Delta)^s U = 0 \quad \text{for } x \in \mathbb{R}^N \text{ and } t > 0. \tag{2.19}$$

This problem has been studied, mainly in probability ([7, 18]), see also [100], and many results are known. When considering initial data $U_0 \in L^1(\mathbb{R}^N)$, or more general,

$$U(0,x) = U_0(x) \quad \text{for } x \in \mathbb{R}^N\,, \tag{2.20}$$

the solution of Problem (2.19)-(2.20) has the integral representation

$$U(x,t) = \int_{\mathbb{R}^N} K_s(x-z,t) U_0(z)\,dz\,, \tag{2.21}$$

where the kernel K_s has Fourier transform $\widehat{K}_s(\xi,t) = e^{-|\xi|^{2s}t}$. If $s = 1$, the function $K_1(x,t)$ is the Gaussian heat kernel.

2.4.1 The fundamental solution. Further results on the asymptotics for large $|x|$

We need some detailed information on the behaviour of the kernel $K_s(x,t)$ for $0 < s < 1$. In the particular case $s = 1/2$, the kernel is explicit, given by the formula

$$K_{1/2}(x,t) = C_N t(|x|^2 + t^2)^{-(N+1)/2}.$$

In general, we know that the kernel $K_s(x,t)$ is the fundamental solution of Problem (2.19), that is $K_s(x,t)$ solves the problem with initial data the Delta function

$$\lim_{t\to 0} K_s(x,t) = \delta(x).$$

It is known that the kernel K_s has the form

$$K_s(x,t) = t^{-N/2s} f(|x|t^{-1/2s})$$

for some profile function, $f(r)$, that is positive and decreasing, and behaves at infinity like $f(r) \sim r^{-(N+2s)}$, cf. [24].

We perform now a further analysis of the properties of the fundamental solution. Our aim is to prove the following result.

Proposition 2.4.1. *For every $s \in (0,1)$, the fundamental solution $K_s(x,t)$ of Problem* (2.19) *is a increasing function in time*

$$\frac{\partial}{\partial t} K_s(x,t) \geq 0 \quad \textit{for all large values of} \;\; |x|/t^{1/2s}.$$

This property is known to be satisfied for the fundamental solution of various types of diffusion equations of evolution type: the Gaussian profile for the Heat Equation, the Barenblatt solution for the Fast Diffusion Equation.

The analysis of the derivative $\dfrac{\partial}{\partial t} K_s(x,t)$ involves not only the characterization of the profile f for large r, but also a similar property for the derivative f'. In fact, we will prove that $f(r)$ and $rf'(r)$ have the same behaviour for large arguments. This is due to the power decay property of the profile f.

We recall that this property is clearly true in the explicit case $s = 1/2$ where $f(s) = (1+s^2)^{-(N+1)/2}$. But it is not true in the limit $s \to 1$, i. e., in the case of the Gaussian profile of the Heat Equation $G(s) = e^{-s^2/4}$. Indeed, we can not obtain the same behaviour for $G(s)$ and $sG'(s)$ since in this case the profile has an exponential expression.

Proof of the proposition. We recall that

$$K_s(x,t) = t^{-\frac{N}{2s}} f_{2s}(1, t^{-\frac{1}{2s}}|x|) \tag{2.22}$$

([24]), where $f_{2s}(1,x)$ is a continuous strictly positive function on $\mathbb{R}^N$ of radial type, which is explicitly given by the expression

$$\begin{aligned} f_{2s}(1,x) &= \left[(2\pi)^{N/2}|x|^{\frac{N}{2}-1}\right]^{-1}\int_0^\infty e^{-\omega^{2s}}\omega^{\frac{N}{2}}J_\nu(|x|\omega)d\omega \\ &= \frac{1}{(2\pi)^{N/2}|x|^N}\int_0^\infty e^{-\left(\frac{\omega}{|x|}\right)^{2s}}\omega^{\frac{N}{2}}J_\nu(\omega)d\omega, \quad \nu=(N-2)/2, \end{aligned}$$

where J_μ denotes the Bessel function of first kind of order μ. For simplicity, we denote $f(r)=f_{2s}(1,x),\ r=|x|$ since $f_{2s}(1,\cdot)$ is a radial function:

$$f(r)=\frac{1}{(2\pi)^{N/2}}\,r^{-N}\int_0^\infty e^{-\left(\frac{\omega}{r}\right)^{2s}}\omega^{\frac{N}{2}}J_\nu(\omega)d\omega, \quad \nu=(N-2)/2. \tag{2.23}$$

Next, we prove an intermediate result, concerning the behaviour of the derivative f'.

Lemma 2.4.1. *Let $s\in(0,1)$ and let $f(r)=f_{2s}(1,x)$ be defined by* (2.23)*. Then*

$$\lim_{r\to\infty} r^{N+2s}(Nf(r)+rf'(r)) = -s^2 2^{2s+1}\frac{1}{\pi^{1+N/2}}(\sin\pi s)\Gamma(s)\Gamma\left(s+\frac{N}{2}\right).$$

In particular, we prove that $rf'(r)\sim -r^{-(N+2s)}$ for large r.

Proof. We compute the derivative with respect to r

$$f'(r)=\frac{1}{(2\pi)^{N/2}}\,r^{-N-1}\int_0^\infty\left(-N+2s\left(\frac{\omega}{r}\right)^{2s}\right)e^{-\left(\frac{\omega}{r}\right)^{2s}}\omega^{\frac{N}{2}}J_\nu(\omega)d\omega.$$

Therefore

$$rf'(r)=-Nf(r)+\frac{1}{(2\pi)^{N/2}}\,r^{-N}\int_0^\infty 2s\left(\frac{\omega}{r}\right)^{2s}\,e^{-\left(\frac{\omega}{r}\right)^{2s}}\omega^{\frac{N}{2}}J_\nu(\omega)d\omega=(I)+(II),$$

where (I)$=-Nf(r)$, and (II) is given by

$$(II)=2s\frac{1}{(2\pi)^{N/2}}\,r^{-(N+2s)}\int_0^\infty e^{-\left(\frac{\omega}{r}\right)^{2s}}\omega^{2s+\frac{N}{2}}J_\nu(\omega)d\omega\,.$$

According to formula (2.29), we can write

$$\omega J_{\frac{N}{2}-1}(\omega)=NJ_{\frac{N}{2}}(\omega)-\omega J_{\frac{N}{2}+1}(\omega),$$

and therefore

$$\begin{aligned} (II) &= 2Ns\frac{1}{(2\pi)^{N/2}}\,r^{-(N+2s)}\int_0^\infty e^{-\left(\frac{\omega}{r}\right)^{2s}}\omega^{2s+\frac{N}{2}-1}J_{\frac{N}{2}}(\omega)d\omega \\ &\quad -2s\frac{1}{(2\pi)^{N/2}}\,r^{-(N+2s)}\int_0^\infty e^{-\left(\frac{\omega}{r}\right)^{2s}}\omega^{2s+\frac{N}{2}}J_{\frac{N}{2}+1}(\omega)d\omega. \end{aligned}$$

Then, according to Pólya (see Blumenthal [24])

$$\lim_{r\to\infty}\int_0^\infty e^{-\left(\frac{\omega}{r}\right)^{2s}}\omega^{2s+\frac{N}{2}-1}J_{\frac{N}{2}}(\omega)d\omega=\frac{2}{\pi}\sin\pi s\int_0^\infty \omega^{2s+\frac{N}{2}-1}K_{\frac{N}{2}}(\omega)d\omega$$

and

$$\lim_{r\to\infty}\int_0^\infty e^{-\left(\frac{\omega}{r}\right)^{2s}}\omega^{2s+\frac{N}{2}}J_{\frac{N}{2}+1}(\omega)d\omega=\frac{2}{\pi}\sin\pi s\int_0^\infty \omega^{2s+\frac{N}{2}}K_{\frac{N}{2}+1}(\omega)d\omega.$$

Here the functions K_μ are described in the paper of Erdélyi [57] (not to be confused with $K_s(x,t)$). Moreover ([57] page 51) we have

$$L_1=\int_0^\infty \omega^{2s+\frac{N}{2}-1}K_{\frac{N}{2}}(\omega)d\omega=2^{2s+\frac{N}{2}-2}\Gamma\left(s+\frac{N}{2}\right)\Gamma(s),$$

$$L_2=\int_0^\infty \omega^{2s+\frac{N}{2}}K_{\frac{N}{2}+1}(\omega)d\omega=2^{2s+\frac{N}{2}-1}\Gamma\left(s+\frac{N}{2}+1\right)\Gamma(s).$$

Therefore,

$$\lim_{r\to\infty}r^{N+2s}\big(rf'(r)+Nf(r)\big)=-2sC_1(N,s),$$

where

$$C_1(N,s):=s2^{2s}\frac{1}{\pi^{1+N/2}}(\sin\pi s)\Gamma(s)\Gamma\left(s+\frac{N}{2}\right). \tag{2.24}$$

If we write this result as

$$r^{N-1}\big(rf'(r)+Nf(r)\big)\sim-2sC_1(N,s)r^{-2s-1},$$

by integrating we obtain $r^Nf(r)\sim C_1(N,s)r^{-2s}$, that is

$$f(r)\sim C_1(N,s)r^{-(N+2s)},$$

which is exactly the result proved in [24]. Moreover, we obtain that

$$\lim_{r\to\infty}r^{N+2s}rf'(r)=-(N+2s)C_1(N,s),$$

that is

$$rf'(r)\sim-r^{-(N+2s)}\quad\text{for large } r.$$

□

We complete the proof of Proposition 2.4.1 on the behaviour of the fundamental solution for large values of $\eta=|x|\,t^{-1/2s}$.

Proof. (of Proposition 2.4.1) The Fundamental solution is given by

$$K_s(x,t)=t^{-\frac{N}{2s}}f(t^{-\frac{1}{2s}}|x|).$$

We compute the derivative in the t variable. According to the scaling formula (2.22) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}K_s(x,t) &= -\frac{N}{2s}t^{-\frac{N}{2s}-1}f(t^{-\frac{1}{2s}}|x|) - \frac{1}{2s}t^{-\frac{N}{2s}-\frac{1}{2s}-1}|x|f'(t^{-\frac{1}{2s}}|x|)\\ &= -\frac{1}{2s}t^{-\frac{N}{2s}-1}\left[Nf(\eta)+\eta f'(\eta)\right], \quad \eta = t^{-\frac{1}{2s}}|x|.\end{aligned}$$

By Lemma 2.4.1 we know that

$$Nf(\eta)+\eta f'(\eta) \sim -2sC_1(N,s)\eta^{-(N+2s)}, \quad \text{for large } \eta,$$

where $C_1(N,s)$ is a positive constant given by formula (2.24). Therefore,

$$\frac{\partial}{\partial t}K_s(x,t) \sim t^{-\frac{N}{2s}-1}C_1(N,s)\eta^{-(N+2s)} = C_1(N,s)|x|^{-(N+2s)} \quad \text{for large } \eta.$$

□

2.4.2 Self-similar solutions of the linear diffusion problem

We study the existence, uniqueness and properties of self-similar solutions of the form

$$U(x,t) = t^{\alpha_1}F(t^{\beta_1}|x|) \tag{2.25}$$

of the linear problem

$$\begin{cases} U_t + (-\Delta)^s U = 0 & \text{for } x\in\mathbb{R}^N \text{ and } t>0,\\ U(0,x) = U_0(x) = C\,|x|^{\gamma}. & \text{for } x\in\mathbb{R}^N, \end{cases} \tag{2.26}$$

where $C>0$, and $0<\gamma<2s$ is given. The constants $\alpha_1,\beta_1\in\mathbb{R}$ will be determined such that $U(x,t)$ is a self-similar solution of Problem (2.26).

Existence of a solution U to Problem (2.26) follows from the representation formula (2.21) since $K_s(x-z,t)u_0(z) \sim |z|^{-(N+2s-\gamma)}$ for large $|z|$, where $\gamma<2s$, and then $K_s(x-z,t)u_0(z)$ is integrable away from the origin.

Let $\eta = t^{\beta_1}|x|$. Then,

$$U_t(x,t) = \alpha_1 t^{\alpha_1-1}F(\eta) + \beta_1 t^{\alpha_1-1}\eta F'(\eta),$$

$$(-\Delta)^s U(x,t) = t^{\alpha_1}(-\Delta)^s(F(t^{\beta_1}|x|)) = t^{\alpha_1}t^{2\beta_1 s}(-\Delta)^s F(\eta).$$

We obtain a first relation on the parameters: $\alpha_1 - 1 = \alpha_1 + 2\beta_1 s$, and then $\beta_1 = -\frac{1}{2s}$.

Equation. The profile F satisfies the equation

$$\alpha_1 F(\eta) + \beta_1\eta F'(\eta) + (-\Delta)^s F(\eta) = 0.$$

Self-similarity condition. The equation is invariant under transformations of the form
$$T_\lambda U(x,t) = \lambda^{-\alpha_1} U(\lambda^{-\beta_1} x, \lambda t).$$
Therefore we impose $U = T_\lambda U$. We apply this to the initial data
$$T_\lambda U(x,0) = \lambda^{-\alpha_1} U(\lambda^{-\beta_1} x, 0) = \lambda^{-\alpha_1 - \beta_1 \gamma} |x|^\gamma$$
and then $\alpha_1 = -\gamma\beta_1$. We obtain the exact value of the similarity exponents
$$\alpha_1 = \frac{\gamma}{2s}, \quad \beta_1 = -\frac{1}{2s}. \tag{2.27}$$
Notice that $\alpha_1 > 0$ and $\beta_1 < 0$. As a solution of the linear problem (2.26), $U(x,t)$ can be computed as a convolution with the kernel $K_s(\cdot,t)$
$$U(x,t) = (K_s(\cdot,t) \star U_0)(x) = \int_{\mathbb{R}^N} K_s(y,t) U_0(x-y) dy.$$
Since the initial data is a radial function $U_0(x) = |x|^\gamma$, then by the properties of the kernel K_s, U will also be a radial function, and therefore the profile F is radial.

Lemma 2.4.2 (Properties of the profile). *The profile F is monotone non-decreasing and it satisfies $\eta F' \leq c_2 F$, for all $\eta \geq 0$.*

Proof. **I. Monotonicity property.** In order to prove the positivity of F we will make use of the Alexandrov Symmetry Principle and we prove that $U(x,t)$ is radially non-decreasing in the space variable $x \in \mathbb{R}^N$.

We start with non-decreasing radial initial data $U_0(x) = |x|^\gamma$. We approximate U_0 with a sequence of radially symmetric and bounded functions $U_{0n} \in L^\infty(\mathbb{R}^N)$ such that $U_{0n}(r) \to C\, n^\gamma$ as $r \to \infty$ and $v_{0n}(r) = C\, n^\gamma - U_{0n}(r) \in L^1(\mathbb{R}^N)$. Let v_n the solution of Problem (2.26) with initial datum v_{0n}. We may apply the Alexandrov Symmetry Principle (that we explain in detail below) to v_n to conclude that it is radially symmetric and non-increasing w.r.t. the space variable. We then put $U_n(x,t) = C\, n^\gamma - v_n(x,t)$, which is radially symmetric and increasing, and solves (2.26) with initial datum U_{0n}. We pass now to the limit $n \to \infty$ to get the same conclusion for U.

Applying the Alexandrov Symmetry Principle. We fix two points x and x' in $\mathbb{R}^N$ such that $|x| < |x'|$. Let H denote the hyperplane perpendicular on the line xx'. Let Ω_1 and Ω_2 be the two sets delimited by the hyperplane H such that the origin is contained in Ω_1. Let Π the symmetry with respect to H that maps Ω_1 into Ω_2. Clearly, $\Pi(x) = x'$, $x \in \Omega_1$. Then one can prove that for every $y \in \Omega_1$ $|y| < |y'|$, where $y' = \Pi(y)$. Since v_{0n} is radially non-increasing, we get that $v_{0n}(y) \geq v_{0n}(\Pi(y))$, for all $y \in \Omega_1$. By applying the Alexandrov Symmetry Principle stated in Theorem 2.5.1 we obtain that $v_n(x) \geq v_n(x')$. The arguments we used can be done for every pair of points $|x| < |x'|$, therefore v_n is radially increasing.

II. Decay at infinity. This follows from the initial data of the solution U. In fact, fixing x and letting $t \to 0$ we get $U(x,t) \to U_0(x) = C|x|^\gamma$ as $t \to 0$, which can be written as $t^{\gamma/2s}|F(t^{-1/2s}x) - C((t^{-1/2s}|x|)^\gamma| \to 0$ as $t \to 0$. In other words, $F(\eta)/\eta^\gamma \to C$ as $\eta \to \infty$.

This characterization of the profile F gives us the following spatial decay for $U(x,t)$ for large times

$$C_2|x|^\gamma \le U(x,t) \le C_1|x|^\gamma \quad \text{for large } |x|t^{-1/2s}.$$

Moreover, we will prove the following relation between F' and F:

$$|\gamma F(\eta) - \eta F'(\eta)| \le \eta^\gamma \quad \text{for large } \eta > 0.$$

As a consequence we can characterize the derivative U_t:

$$U_t(x,t) = t^{\alpha_1 - 1}\frac{1}{2s}\left(\gamma F(\eta) - \eta F'(\eta)\right), \quad \eta = t^{-1/2s}|x|,$$

$$U_t(x,t) \sim t^{-1}|x|^\gamma \quad \text{for large values of } \ t^{-1/2s}|x|.$$

The first step will be to obtain a formula for the profile $F(\eta)$. Therefore

$$\begin{aligned} U(x,t) = K_s(x,t) \star U_0(x) &= t^{-\frac{N}{2s}} \int_{\mathbb{R}^N} f(t^{-\frac{1}{2s}}|x-y|)|y|^\gamma dy, \quad z = t^{-\frac{1}{2s}}y \\ &= t^{\frac{\gamma}{2s}} \int_{\mathbb{R}^N} f(t^{-\frac{1}{2s}}x - z)|z|^\gamma dz. \end{aligned}$$

Since $U(x,t)$ has the self similar form (2.25) then

$$F(t^{-\frac{1}{2s}}x) = \int_{\mathbb{R}^N} f(t^{-\frac{1}{2s}}x - z)|z|^\gamma dz = (f \star U_0)(t^{-\frac{1}{2s}}x), \quad \forall x \in \mathbb{R}^N, t > 0,$$

that is

$$F(\eta) = \int_{\mathbb{R}^N} f(\eta - z)|z|^\gamma dz, \quad \forall \eta \in \mathbb{R}^N.$$

Let us continue using the notations

$$F(|\eta|) = F(\eta), \quad f(|\eta|) = f(\eta).$$

We fix $\eta \in \mathbb{R}^N$. Let $|\eta| = \bar\eta$ and $\eta = \bar\eta e$ for a vector $e \in \mathbb{R}^N$ with $|e| = 1$. Then

$$F(\bar\eta) = \int_{\mathbb{R}^N} f(|z|)|\eta - z|^\gamma dz = \bar\eta^{N+\gamma} \int_{\mathbb{R}^N} f(|\bar\eta y|)|e - y|^\gamma dy, \quad z = \bar\eta y.$$

We differentiate in $\bar\eta$

$$F'(\bar\eta) = \bar\eta^{N+\gamma-1} \int_{\mathbb{R}^N} \left[(N+\gamma)f(|\bar\eta y|) + \bar\eta y f'(|\bar\eta y|)\right] |e - y|^\gamma dy.$$

Therefore

$$\bar\eta F'(\bar\eta) - \gamma F(\bar\eta) = \bar\eta^{N+\gamma} \int_{\mathbb{R}^N} \left[Nf(|\bar\eta y|) + \bar\eta y f'(|\bar\eta y|) \right] |e-y|^\gamma dy, \quad z = \bar\eta y$$

$$\bar\eta F'(\bar\eta) - \gamma F(\bar\eta) = \bar\eta^{\gamma} \int_{\mathbb{R}^N} \left[Nf(|z|) + z f'(|z|) \right] \left| e - \frac{z}{\bar\eta} \right|^\gamma dz.$$

We know that $Nf(r) + rf'(r) \sim -C_1 r^{-(N+2s)}$ for large r. Since we deal with a convolution, we will use the information only in the sense of modulus. We fix $R > 0$ such that

$$C_1 r^{-(N+2s)} \le |Nf(r) + zf'(r)| \le C_2 r^{-(N+2s)}, \quad \forall r \ge R.$$

The values of R, C_1, C_2 depend on the profile f of the heat kernel $K_s(x,t)$. They are independent of the variable η used in this proof. Then,

$$\bar\eta^{-\gamma} \left| \bar\eta F'(\bar\eta) - \gamma F(\bar\eta) \right| \le \int_{\mathbb{R}^N} \left| Nf(|z|) + zf'(|z|) \right| \cdot \left| e - \frac{z}{\bar\eta} \right|^\gamma dz = I + II,$$

where

$$I = \int_{|z|\le R} \left| Nf(|z|) + zf'(|z|) \right| \cdot \left| e - \frac{z}{\bar\eta} \right|^\gamma dz \le C \int_{|z|\le R} \left| e - \frac{z}{\bar\eta} \right|^\gamma dz \le C \left(1 + \frac{R}{\bar\eta} \right)^\gamma R^N.$$

The second term is estimated as follows:

$$\begin{aligned}
II &= \int_{|z|\ge R} \left| Nf(|z|) + zf'(|z|) \right| \cdot \left| e - \frac{z}{\bar\eta} \right|^\gamma dz \le C_2 \int_{|z|\ge R} |z|^{-(N+2s)} \left| e - \frac{z}{\bar\eta} \right|^\gamma dz \\
&\le C_2 \int_{|z|\ge R} |z|^{-(N+2s)} \left(1 + \frac{|z|}{\bar\eta} \right)^\gamma \le C_2 \int_{|z|\ge R} |z|^{-(N+2s)} \left(2^\gamma + (2|z|/\bar\eta)^\gamma \right) dz \\
&= 2^\gamma C_2 \int_{|z|\ge R} |z|^{-(N+2s)} dz + 2^\gamma C_2 \, \bar\eta^{-\gamma} \int_{|z|\ge R} |z|^{-(N+2s)+\gamma} dz, \quad \text{we know } \gamma < 2s, \\
&= C_3 R^{-2s} + C_3 \frac{1}{\bar\eta^\gamma} R^{\gamma-2s}, \quad \text{where} \quad C_3 = C_3(C_2, \gamma, \mathrm{meas}(\partial B_1)) > 0.
\end{aligned}$$

We conclude that

$$I + II \le C_4 + C_5 \frac{1}{\bar\eta^\gamma},$$

where the constants C_4 and C_5 depend on R, γ, C_1, C_2. Now, recall that $|\bar\eta F'(\bar\eta) - \gamma F(\bar\eta)| \le \bar\eta^\gamma (I + II)$. Therefore we have proved that

$$\bar\eta^{-\gamma} \left| \bar\eta F'(\bar\eta) - \gamma F(\bar\eta) \right| \le C_4 + C_5 \quad \text{for large } \bar\eta \ge 1.$$

□

2.5 Appendix B

2.5.1 Alexandrov Reflection Principle

We recall the version of Alexandrov's symmetry principle that holds in the case of the nonlinear parabolic problem

$$u_t = (-\Delta)^s u^m, \quad u(0,x) = u_0(x), \tag{2.28}$$

posed in $\mathbb{R}^N$, with $(-\Delta)^s = (-\Delta)^s$, $m > 0$, $u_0 \in L^1(\mathbb{R}^N)$. Let us take a hyperplane H that divides $\mathbb{R}^N$ into two half-spaces Ω_1 and Ω_2 and consider the symmetry Π with respect to H that maps Ω_1 into Ω_2. The following result is proved as Theorem 15.2 in [107]:

Theorem 2.5.1. *Let u be the unique solution of Problem* (2.28) *with initial data u_0. Under the assumption that*

$$u_0(x) \geq u_0(\Pi(x)) \quad \text{in } \Omega_1$$

we have that for all $t > 0$

$$u(x,t) \geq u(\Pi(x),t) \quad \text{for } x \in \Omega_1.$$

2.5.2 Bessel functions of first kind

The Bessel function J_μ of first kind can be introduced through a series expansion, cf. [1],

$$J_\mu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\mu+1)} \left(\frac{z}{2}\right)^{2k+\mu}.$$

We mention the following recurrence formulas:

$$J'_\mu(z) = \frac{1}{2}\left(J_{\mu-1}(z) - J_{\mu+1}(z)\right), \quad \text{for } \mu \neq 0.$$

$$J'_0(z) = -J_1(z),$$

$$J_\mu(z) = \frac{z}{2\mu}\left(J_{\mu-1}(z) + J_{\mu+1}(z)\right). \tag{2.29}$$

$$\int_0^\infty K_a(t) t^{b-1} dt = 2^{b-2}\Gamma\left(\frac{b+a}{2}\right)\Gamma\left(\frac{b-a}{2}\right), \quad \mathrm{Re}(b \pm a) > 0. \tag{2.30}$$

The *modified Hankel functions*, cf. [57] page 82, are defined by

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)\,dt, \quad \text{for } \mathrm{Re}(z) > 0. \tag{2.31}$$

Then $K_\nu(z) \in \mathbb{R}$ when $\nu \in \mathbb{R}$ is real and $z \in \mathbb{R}_+$. When $\nu = n + \frac{1}{2}$, $n \in \mathbb{N}$ then

$$K_{n+1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{m=0}^{n} (2z)^{-m} \frac{\Gamma(n+m+1)}{m!\,\Gamma(n+1-m)}.$$

2.5.3 The Fractional Laplacian operator. Functional Settings

Let $s \in (0,1)$. Let $\mathcal{F}$ denote the Fourier transform. We consider

$$H^s(\mathbb{R}^N) = \{u : L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty\}$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi.$$

For function $u \in H^s$, the Fractional Laplacian is defined by

$$(-\Delta)^s u(x) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = C\mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)),$$

where $C_{N,s} = \pi^{-(2s+N/2)}\Gamma(N/2+s)/\Gamma(-s)$.

For an interesting introduction to the Fractional Laplacian operator we refer to Valdinoci [100]. For the functional setting we refer to Di Nezzza, Palatucci and Valdinoci [54].

2.5.4 Functional inequalities related to the fractional Laplacian

We recall some functional inequalities related to the fractional Laplacian operator that we used throughout the paper. We refer to [47].

Lemma 2.5.1 (Stroock-Varopoulos Inequality). *Let $0 < s < 1$, $r > 1$. Then*

$$\int_{\mathbb{R}^N} |v|^{r-2} v (-\Delta)^s v dx \geq \frac{4(q-1)}{q^2} \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} |v|^{r/2} \right|^2 dx \tag{2.32}$$

for all $v \in L^q(\mathbb{R}^N)$ such that $(-\Delta)^s v \in L^q(\mathbb{R}^N)$.

Lemma 2.5.2 (Generalized Stroock-Varopoulos Inequality). *Let $0 < s < 1$. Then*

$$\int_{\mathbb{R}^N} \psi(v) (-\Delta)^s v dx \geq \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} \Psi(v) \right|^2 dx \tag{2.33}$$

whenever $\psi' = (\Psi')^2$.

Theorem 2.5.2 (Sobolev Inequality). *Let $0 < s < 1$ and $2s < N$. Then*

$$\|f\|_{\frac{2N}{N-2s}} \leq \mathcal{S}_s \left\| (-\Delta)^{s/2} f \right\|_2, \tag{2.34}$$

where the best constant is computed in [28] page 31.

Chapter 3

The Fisher-KPP equation with nonlinear fractional diffusion

In this chapter we study the propagation properties of nonnegative and bounded solutions of the class of reaction-diffusion equations $u_t + (-\Delta)^s(u^m) = f(u)$. This work is developed in the paper [97].

We consider the following reaction-diffusion problem

$$\begin{cases} u_t(x,t) + (-\Delta)^s u^m(x,t) = f(u) & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{KPP}$$

where $(-\Delta)^s$ is the Fractional Laplacian operator with $s \in (0,1)$. We are interested in studying the propagation properties of nonnegative and bounded solutions of this problem in the spirit of the Fisher-KPP theory. Therefore, we assume that the reaction term $f(u)$ satisfies

$$f \in C^1([0,1]) \text{ is a concave function with } f(0) = f(1) = 0, \quad f'(1) < 0 < f'(0). \tag{3.1}$$

For example we can take $f(u) = u(1-u)$. Our results will depend on the parameters m and s, according to the ranges $m_c < m < m_1$, $m_1 < m \le 1$, and $m > 1$, where

$$m_c = \frac{(N-2s)_+}{N}, \qquad m_1 = \frac{N}{N+2s}\,.$$

Concept of solution to Problem (KPP)

According to [47] there exists a unique mild solution of Problem (2.1) corresponding to the initial datum $u_0 \in L^1(\mathbb{R}^N)$, $0 \le u_0 \le 1$, constructed by means of the tools of semigroup theory. Moreover, such u is in fact a strong solution of the equation. In the case $m > 1$, the classical regularity of the solution follows from [11], and this has been extended to $m < 1$ up to the extinction time (if there is one). Quantitative estimates of positivity of the solution for any $m > 0$ corresponding to non-negative data have been proved in [28]. Recently, classical regularity of strong solutions was proved in [48].

As a consequence, one obtains by rather standard methods the existence, uniqueness and regularity properties of the solution to Problem (KPP) corresponding to the initial datum $u_0 \in L^1(\mathbb{R}^N)$, $0 \le u_0 \le 1$. In order to prove the existence of a solution of the problem $u_t + (-\Delta)^s u^m = f(u)$, since f is Lipschitz. The idea is to prove that the map $u_0 \mapsto u$ is a m-ω-accretive operator. Standard properties, like the maximum principle hold also in our setting.

3.1 Motivation and organization of the results

3.1.1 Perspective. The traveling wave behaviour

The problem with standard diffusion goes back to the work of Kolmogorov, Petrovskii and Piskunov, see [77], that presents the most simple reaction-diffusion equation concerning the concentration u of a single substance in one spatial dimension,

$$\partial_t u = D u_{xx} + f(u). \tag{3.2}$$

The choice $f(u) = u(1-u)$ yields Fisher's equation [60] that was originally used to describe the spreading of biological populations. The celebrated result says that the long-time behaviour of any solution of (3.2), with suitable data $0 \le u_0(x) \le 1$ that decay fast at infinity, resembles a traveling wave with a definite speed. When considering equation (3.2) in dimensions $N \ge 1$, the problem becomes

$$u_t - \Delta u = f(u) \quad \text{in } (0,+\infty) \times \mathbb{R}^N, \tag{3.3}$$

which corresponds to (KPP) in the case when $(-\Delta)^s = -\Delta$, the standard Laplacian. This case has been studied by Aronson and Weinberger in [9, 10], where they prove the following result.

Theorem AW. *Let u be a solution of (3.3) with $u_0 \neq 0$ compactly supported in $\mathbb{R}^N$ and satisfying $0 \le u_0(\cdot) \le 1$. Let $c_* = 2\sqrt{f'(0)}$. Then,*

1. *if $c > c_*$, then $u(x,t) \to 0$ uniformly in $\{|x| \ge ct\}$ as $t \to \infty$.*
2. *if $c < c_*$, then $u(x,t) \to 1$ uniformly in $\{|x| \le ct\}$ as $t \to \infty$.*

In addition, problem (3.3) admits planar traveling wave solutions connecting 0 and 1, that is, solutions of the form $u(x,t) = \phi(x \cdot e + ct)$ with

$$-\phi'' + c\phi' = f(\phi) \text{ in } \mathbb{R}, \quad \phi(-\infty) = 0, \ \phi(+\infty) = 1.$$

This asymptotic traveling-wave behaviour has been generalized in many interesting ways. Of concern here is the consideration of nonlinear diffusion. De Pablo and Vázquez study in [50] the existence of traveling wave solutions and the property of finite propagation for the reaction-diffusion equation

$$u_t = (u^m)_{xx} + \lambda u^n(1-u), \quad (x,t) \in \mathbb{R} \times (0,\infty)$$

with $m > 1$, $\lambda > 0$, $n \in \mathbb{R}$ and $u = u(x,t) \geq 0$. Similar results hold also for other slow diffusion cases, $m > 1$, studied by de Pablo and Sánchez ([49]).

3.1.2 Non-traveling wave behaviour

Departing from these results, King and McCabe examined in [75] a case of fast diffusion, namely

$$u_t = \Delta u^m + u(1-u), \quad x \in \mathbb{R}^N, t > 0,$$

where $(N-2)_+/N < m < 1$. They showed that the problem does not admit traveling wave solutions. Using a detailed formal analysis, they also showed that level sets of the solutions of the initial-value problem with suitable initial data propagate exponentially fast in time. They extended the results to all $0 < m < 1$.

On the other hand, and independently, Cabré and Roquejoffre in [35, 37] studied the case of fractional linear diffusion, $s \in (0,1)$ and $m = 1$, and they concluded in the same vein that there is no traveling wave behaviour as $t \to \infty$, and indeed the level sets propagate exponentially fast in time. The fast propagation is not surprising because of the long distance dispersal, even if the diffusion is linear.

Motivated by these two examples of break of the asymptotic TW structure, we study here the case of a diffusion that is both fractional and nonlinear, namely problem (KPP) in the range $s \in (0,1)$ and $m > m_c$. The initial datum $u_0(x) : \mathbb{R}^N \to [0,1]$ and satisfies a growth condition of the form

$$0 \leq u_0(x) \leq C|x|^{-\lambda(N,s,m)}, \quad \forall x \in \mathbb{R}^N, \tag{3.4}$$

where the exponent $\lambda(N,s,m)$ is stated explicitly in the different ranges, $m_c < m < m_1$ and $m_1 < m$. In this paper we establish the negative result about traveling wave behaviour, more precisely, we prove that an exponential rate of propagation of level sets is true in all cases. We also explain the mechanism for it in simple terms: the exponential rate of propagation of the level sets of solutions (with initial data having a certain minimum decay for large $|x|$) is a consequence of the power-like decay behaviour of the fundamental solutions of the diffusion problem studied in [107]. Therefore, we obtain two main cases in the analysis, $m_c < m < m_1$ and $m > m_1$, depending on that behaviour.

3.1.3 Main results

The existence of a unique mild solution of problem (KPP) follows by semigroup approach. The mild solution corresponding to an initial datum $u_0 \in L^1(\mathbb{R}^N)$, $0 \le u_0 \le 1$ is in fact a positive, bounded, strong solution with classical regularity. In the Appendix we give a brief discussion of these properties. Let us introduce some notations. Throughout the paper we will consider $m > m_c$. Once and for all, we put

$$\beta = 1/(N(m-1) + 2s),$$

$$\sigma_1 = \frac{1-m}{2s} f'(0), \quad \sigma_2 = \frac{1}{N+2s} f'(0), \quad \sigma_3 = \frac{1+2(m-1)\beta s}{N+2s} f'(0). \tag{3.5}$$

The value σ_1 appears for $m_c < m < m_1$ and then $\sigma_1 > \sigma_2$. Notice also that $\sigma_2 < \sigma_3$ for $m > 1$. Here is the precise statement of our main results for the solutions of the generalized KPP problem (KPP).

Theorem 3.1.1. *Let $N \ge 1$, $s \in (0,1)$, f satisfying* (3.1) *and $m_1 < m \le 1$. Let u be a solution of* (KPP)*, where $0 \le u_0(\cdot) \le 1$ is measurable, $u_0 \not\equiv 0$ and satisfies*

$$0 \le u_0(x) \le C|x|^{-(N+2s)}, \quad \forall x \in \mathbb{R}^N. \tag{3.6}$$

Then:

1. *if $\sigma > \sigma_2$, then $u(x,t) \to 0$ uniformly in $\{|x| \ge e^{\sigma t}\}$ as $t \to \infty$;*
2. *if $\sigma < \sigma_2$, then $u(x,t) \to 1$ uniformly in $\{|x| \le e^{\sigma t}\}$ as $t \to \infty$.*

Theorem 3.1.2. *Let $N \ge 1$, $s \in (0,1)$, f satisfying* (3.1) *and $m_c < m < m_1$. Let u be a solution of* (KPP)*, where $0 \le u_0(\cdot) \le 1$ is measurable, $u_0 \not\equiv 0$ and satisfies*

$$0 \le u_0(x) \le C|x|^{-2s/(1-m)}, \quad \forall x \in \mathbb{R}^N. \tag{3.7}$$

Then:

1. *if $\sigma > \sigma_1$, then $u(x,t) \to 0$ uniformly in $\{|x| \ge e^{\sigma t}\}$ as $t \to \infty$;*
2. *if $\sigma < \sigma_1$, then $u(x,t) \to 1$ uniformly in $\{|x| \le e^{\sigma t}\}$ as $t \to \infty$.*

Theorem 3.1.3. *Let $N \ge 1$, $s \in (0,1)$, f satisfying* (3.1) *and $m > 1$. Let u be a solution of* (KPP)*, where $0 \le u_0(\cdot) \le 1$ is measurable, $u_0 \not\equiv 0$ and satisfies*

$$0 \le u_0(x) \le C|x|^{-(N+2s)}, \quad \forall x \in \mathbb{R}^N.$$

Then:

1. *if $\sigma > \sigma_3$, then $u(x,t) \to 0$ uniformly in $\{|x| \ge e^{\sigma t}\}$ as $t \to \infty$;*
2. *if $\sigma < \sigma_2$, then $u(x,t) \to 1$ uniformly in $\{|x| \le e^{\sigma t}\}$ as $t \to \infty$.*

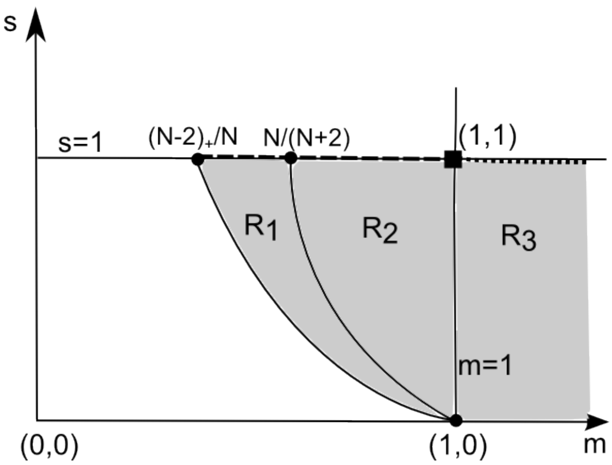


FIGURE 3.1: Ranges of parameters m and s: we study the cases $R_1 = \{s \in (0,1),\ (N-2s)_+/N < m < N/(N+2s)\}$; $R_2 = \{s \in (0,1),\ N/(N+2s) < m \le 1\}$, $R_3 = \{s \in (0,1),\ m > 1\}$

Remarks. In all ranges of parameters $m > m_c$, there appear critical values of σ with an influence on the behaviour of the level sets.

• In the case $m_1 < m < 1$, the case $\sigma = \sigma_2$ is still open. This critical exponent is the same as in the case of the linear diffusion $m = 1$, proved in [37].

• In the range $m_c < m < m_1$, the case $\sigma = \sigma_1$ is still open. In particular, for the classical case $s = 1$ and $f(u) = u(1-u)$ we get $\sigma_1 = \frac{1-m}{2}$, which is a critical speed found by King and McCabe [75]. In this way, we complete their result with rigorous proofs to all $s \in (0,1)$.

• In the case $m > 1$, we do not cover the entire interval $[\sigma_2, \sigma_3]$. If we could prove that the behaviour in this interval is the same as in the case $\sigma > \sigma_3$, then the results of Theorems 3.1.1 and 3.1.3 would agree.

• The result of Theorems 3.1.1 and 3.1.2 is true also in the case $m = m_1$, where $\sigma_1 = \sigma_2$. The outline of the proof is the same, but there are a number of additional technical difficulties, typical of borderline cases. We have decided to skip the lengthy analysis of this case because of the lack of novelty for our intended purpose.

Our main conclusion is that exponential propagation is shown to be the common occurrence, and the existence of traveling wave behaviour is reduced to the classical KPP cases mentioned at the beginning of this discussion (see dotted line in Figure 3.1).

As we have already mentioned, one of the motivations of the work was to make clear the mechanism that explains the exponential rate of expansion in simple terms, even in this situation that is more complicated than [37]. In fact, due to the nonlinearity, the solution of the diffusion problems involved in the proofs does not admit an integral representation as the case $m = 1$. Instead, we will use as an essential tool the behaviour of the fundamental solution of the Fractional Porous Medium Equation, also called Barenblatt solution, recently studied in [107]. To be precise, the decay rate of the tail of these solutions as $|x| \to \infty$ is the essential information we use to calculate the rates of expansion. This information is combined with more or less usual techniques of linearization and comparison with sub- and super-solutions. We also need accurate lower estimates for positive solutions of Fractional Porous Medium Equation, and a further selfsimilar analysis for the linear diffusion problem.

3.1.4 Organization of the proofs

In Section 3.2, under the assumption of initial datum with the decay (3.4), we prove convergence to 0 in the outer set $\{|x| \geq e^{\sigma t}\}$ by constructing a super-solution of the linearized problem with reaction term $f'(0)u$. The arguments hold for σ larger than the corresponding critical velocity.

In Section 3.3 we prove convergence to 1 on the inner sets $\{|x| \leq e^{\sigma t}\}$ in various steps. We only assume $0 \leq u_0 \leq 1$, $u_0 \not\equiv 0$. We first show that the solution reaches a certain minimum profile for positive times, thanks to the analysis of Theorem 2.3.1. We then perform an iterative proof of the conservation in time of this minimum level, and finally convergence to 1 is obtained by constructing a super-solution to the problem satisfied by $1 - u^m$. Therefore, we deal with a problem of the form

$$a(x,t)\, w_t(x,t) + (-\Delta)^s w(x,t) + b_0\, w(x,t) \geq 0.$$

A suitable choice for constructing the super-solution w is represented by self-similar solutions of the form $U(x,t) = t^{\alpha'} F(|x| t^{-\beta'})$ of the linear problem

$$U_t + (-\Delta)^s U = 0 \tag{3.8}$$

with radial increasing initial data. This motivates us to derive a number of properties of the linear diffusion problem (3.8), also known as the Fractional Heat Equation. In particular, we need to show that the profile F mentioned above has the same asymptotic behaviour as the initial data. In order to establish such fact we have to review, Section 2.4, the properties of the fundamental solution of Problem (3.8)

$$K_s(x,t) = t^{-\frac{N}{2s}} f(t^{-\frac{1}{2s}}|x|), \qquad f(r) \sim r^{-(N+2s)}.$$

We perform a further analysis of the profile f by proving that $rf' \sim r^{-(N+2s)}$.

Remark. As a consequence of the exponential propagation of the level sets, we immediately obtain the non-existence of traveling wave solutions of the form $u(x,t) = \varphi(x + t\cdot e)$. However, our results amount to the existence of a kind of logarithmic traveling wave behaviour, that is a kind of wave solutions that travel linearly if we measure distance in a logarithmic scale. This whole issue deserves further investigation.

3.1.5 Quantitative estimates for the fractional diffusion problem

The study of the sub- and super-solutions is strongly determined by the existence of suitable lower parabolic estimates for the associated diffusion problem, the Fractional Porous Medium Equation (FPME). In Chapter 2.3 we recall the properties of the (FPME).

3.1.6 Remarks on the Reaction Problem

(a) As a further evidence of the influence of the tail of the data on the propagation rate, we consider the purely reactive problem (no diffusion)

$$u_t = f(u), \quad x \in \mathbb{R}^N,\ t > 0\,, \tag{3.9}$$

with initial datum u_0 and $f(u) \sim f'(0)u$ as $u \to 0$. It is easy to see that when we simplify $f(u)$ to $f'(0)u$, the exact solution is

$$u(x,t) = u_0 e^{f'(0)t}.$$

Let us examine the level sets in two particular cases.

(b) Exponential decay. By considering initial datum of the form $u_0(x) \sim e^{-x^2}$ for large $|x|$, then the solution $u(x,t)$ satisfies a similar behaviour

$$u(x,t) \sim e^{-(x^2-at)} \text{ for large } x.$$

The level sets $u(x,t) =$ constant are characterized by $x = \sqrt{at+c}$.

(c) Power decay. By considering initial datum of the form $u_0(x) \sim |x|^{-(N+2s)}$ for large $|x|$, then the solution $u(x,t)$ is such that

$$u(x,t) \sim e^{at}|x|^{-(N+2s)}.$$

The level sets $u(x,t) =$ constant are characterized by $|x| \sim e^{\frac{a}{N+2s}t}$.

(d) Formal observation. The fractional diffusion term $(-\Delta)^s u^m$ does not change the basic behaviour of the solution for large $|x|$. From the above observations it follows that, for large $|x|$, the solution of the reaction-diffusion Problem (KPP) behaves like the solution of Problem (3.9), that is, the non-diffusion case, with datum of the form $u_0(x) \sim |x|^{-(N+2s)}$ for large $|x|$. This fact has been also observed by King and McCabe in [75] in the fast diffusion case with the standard Laplace operator.

3.1.7 Comment on applications and mathematical motivation

Anomalous diffusion processes with long range effects connected to Levy flights in stochastic processes are usually modeled with nonlocal operators, in particular with the fractional Laplacian. They describe different phenomena in physics, finance, biology and many others. Equations involving anomalous diffusion may take a nonlinear form (see [28] for a more detailed summary).

The reaction-diffusion problem (KPP) with linear diffusion $m = 1$, recently studied by Cabré and Roquejoffre [37], appears in population dynamics (see [17]). The nonlocal character of the diffusion operator generates the following event: the stable state $u = 1$ invades faster (with exponential speed) the unstable state $u = 0$. This behaviour was seen already in the case of local nonlinear diffusion with nonlinearity $m < 1$ (fast diffusion) by King and McCabe [75]. The study of the problem involving nonlinear fractional diffusion and KPP reaction is motivated by such preceding works. We show that the exponential invasion of the unstable state by the stable one is a quite general phenomenon that holds for a wide range of equation combining nonlinearity and KPP reaction. As a conclusion, the traveling wave with constant speed of the original KPP model looks in that respect like a very special phenomenon.

3.2 Evolution of level sets of solutions to Problem (KPP)

In this section we start the proof of the main result of the paper on evolution of level sets with exponential speed of propagation. In a first step we prove the convergence to zero on outer sets. Since the decay assumption on the initial data is the same for $m_1 < m < 1$ and $m > 1$, we will treat both cases, as well as $m = 1$, in the following lemma.

Lemma 3.2.1. *We consider $m > m_1$ and let u be the solution of Problem* (KPP) *with initial datum $u_0(x) \in L^1(\mathbb{R}^N)$, $0 \le u_0 \le 1$. We assume that u_0 satisfies the decay property*

$$u_0(x) \le C|x|^{-(N+2s)} \quad \text{for all } x \in \mathbb{R}^N. \tag{3.10}$$

Then, for $\sigma > \sigma_3$ if $m > 1$ (respectively, for $\sigma > \sigma_2$ if $m_1 < m \le 1$), we have

$$u(x,t) \to 0 \quad \text{as} \quad t \to \infty \tag{3.11}$$

uniformly for $|x| \ge e^{\sigma t}$.

Proof. We consider the solution $\overline{u}(x,t)$ of the linearized problem

$$\overline{u}_t + (-\Delta)^s \overline{u}^m = f'(0)\overline{u}, \quad \overline{u}(0,x) = u_0(x).$$

Since f is a concave function, we have $f'(0)s \ge f(s) \quad \forall s \in [0,1]$, and thus $\overline{u}$ is a super-solution of Problem (KPP), which implies the upper estimate

$$u(x,t) \le \overline{u}(x,t) \quad \text{for } t \ge 0,\ x \in \mathbb{R}^N.$$

Next, we define $\tilde{v}(x,\tau)$ by

$$\tilde{v}(x,\tau) = e^{-f'(0)t}\overline{u}(x,t), \tag{3.12}$$

and new time

$$\tau = \frac{1}{(m-1)f'(0)}\left[e^{(m-1)f'(0)t} - 1\right] \text{ if } m > 1, \tag{3.13}$$

$$\tau = \frac{1}{(1-m)f'(0)}\left[1 - e^{-(1-m)f'(0)t}\right] \text{ if } m < 1, \tag{3.14}$$

and $\tau = t$ for $m = 1$. It is immediate to check that $\tilde{v}(x,\tau)$ is a solution of the (FPME) with initial datum $\tilde{v}_0 = u_0$. Let $B_M(x,\tau)$ be the Barenblatt solution with mass M of the (FPME), as defined in Section 2.1. By virtue of the properties of the Barenblatt solutions and assumption (3.10) on the initial data, we conclude there exists $M > 0$ big enough and $\tau_0 > 0$ such that

$$\tilde{v}_0(x) \le B_M(x,\tau_0).$$

By the Maximum Principle

$$\tilde{v}(x,\tau) \le B_M(x,\tau+\tau_0), \quad \forall x \in \mathbb{R}^N, t > 0.$$

Now, using the characterization of the decay of the Barenblatt profile given by relation (2.5), we obtain that there exists $K_1 > 0$ such that $F_M(r) \le K_1 r^{-(N+2s)}$ for all $r \ge 0$. We obtain the following upper estimate on the solution u of Problem (KPP):

$$\begin{aligned}
u(x,t) &\le \overline{u}(x,t) = e^{f'(0)t}\tilde{v}(x,\tau) \\
&\le e^{f'(0)t}B_M(x,\tau+\tau_0) = e^{f'(0)t}(\tau+\tau_0)^{-\alpha}F_M(|x|(\tau+\tau_0)^{-\beta}) \\
&\le e^{f'(0)t}(\tau+\tau_0)^{-\alpha}K_1(|x|(\tau+\tau_0)^{-\beta}))^{-(N+2s)} \\
&= K_1 e^{f'(0)t}(\tau+\tau_0)^{2\beta s}|x|^{-(N+2s)}.
\end{aligned}$$

Case $m > 1$. In order to continue the estimate, we remark that for large times t, the term $\tau^{2\beta s}$ has an influence on the result only in the case $m > 1$. Then $(\tau+\tau_0)^{2\beta s} \le e^{(m-1)f'(0)t}$ for large t. Let us assume that $|x| \ge e^{\sigma t}$. Then one has

$$u(x,t) \le CK_1 e^{f'(0)t}\tau^{2\beta s}e^{-\sigma(N+2s)t} = CK_1 e^{[f'(0)+2f'(0)(m-1)\beta s - \sigma(N+2s)]t}.$$

We want to have $f'(0) + 2f'(0)(m-1)\beta s - \sigma(N+2s) < 0$, which is just the condition

$$\sigma > \frac{1+2(m-1)\beta s}{N+2s}f'(0) = \sigma_3.$$

We have obtained the convergence of $u(x,t)$ to 0 as $t \to \infty$, for $|x| \ge e^{\sigma t}$.

Case $m \le 1$. In this case, the term $(\tau+\tau_0)^{2\beta s}$ is bounded for every $t > 0$ as we can see from (3.14). As before, we assume $|x| \ge e^{\sigma t}$. Then, we get

$$u(x,t) \le CK_1 e^{f'(0)t}e^{-\sigma(N+2s)t} = CK_1 e^{[f'(0)-\sigma(N+2s)]t}.$$

For $\sigma > \sigma_2 = \frac{f'(0)}{N+2s}$, the exponent is negative $f'(0) - \sigma(N+2s) < 0$, and we obtain the convergence of $u(x,t)$ to 0 as $t \to \infty$.

□

Lemma 3.2.2. *We consider $m_c < m < m_1$. Let u be the solution of problem* (KPP) *with initial datum $u_0(x) \in L^1(\mathbb{R}^N)$, $0 \le u_0 \le 1$ and we assume u_0 satisfies the decay property*

$$u_0(x) \le C|x|^{-2s/(1-m)} \quad \textit{for all } x \in \mathbb{R}^N.$$

Then, for $\sigma > \sigma_1$ we have

$$u(x,t) \to 0, \quad t \to \infty$$

uniformly for $|x| \ge e^{\sigma t}$.

Proof. The proof follows the same idea as Lemma 3.2.1, since the Barenblatt solution $B_M(x,\tau) = \tau^{-\alpha} F_M(|x|\tau^{-\beta})$ of the diffusion problem satisfies $F_M(r) \sim r^{-2s/(1-m)}$ according to Theorem 2.1.2. Therefore, we obtain the estimate

$$\begin{aligned} u(x,t) &\le e^{f'(0)t}(\tau+\tau_0)^{-\alpha} K_1(|x|(\tau+\tau_0)^{-\beta}))^{-2s/(1-m)} \\ &= K_1 e^{f'(0)t}(\tau+\tau_0)^{1/(1-m)}|x|^{-2s/(1-m)}. \end{aligned}$$

Since $m < 1$, the term $(\tau+\tau_0)^{1/(1-m)}$ is controlled by $e^{f'(0)t}$ and then, for $|x| \ge e^{\sigma t}$ we obtain

$$u(x,t) \le K_1 e^{f'(0)t - 2s\sigma t/(1-m)}.$$

For $\sigma > \frac{1-m}{2s} f'(0) = \sigma_1$ we obtain the desired convergence to 0 as $t \to \infty$.

□

Remarks

I. When $m = 1$ we recover the minimal speed $\sigma_2 = f'(0)/(N+2s)$ obtained by Cabré and Roquejoffre in [37]. The proof is similar, but in the nonlinear case we have to make an exponential change of time variable. Note also that we only use the decay properties of the fundamental solution.

II. The value of the critical exponent σ_2 can be easily obtained from the following formal study of the level lines of $u(x,t)$. Thus, the set $\{u(x,t) \sim \epsilon\}$ can be written in terms of $\tilde{v}(x,\tau)$ defined in (3.12) as

$$e^{f'(0)t}\tilde{v}(x,\tau) \sim \epsilon. \tag{3.15}$$

By Theorem 2.1.3, $\tilde{v}(x,\tau)$ behaves like the Barenblatt solution of the Fractional Porous Medium Equation (2.1) (we discuss only the case $m > m_1$):

$$\tilde{v}(x,\tau) \sim B(x,\tau) = \tau^{-\alpha} F(r), \quad F(r) \sim r^{-(N+2s)}, \quad r = |x|\tau^{-\beta}.$$

From [107], we know that $B(x,\tau) \sim \tau^{-\alpha+\beta(N+2s)}|x|^{-(N+2s)}$, thus $\tilde{v}(x,\tau) \sim \tau^{2\beta s}|x|^{-(N+2s)}$. At this moment, (3.15) implies $e^{f'(0)t}\tau^{2\beta s}|x|^{-(N+2s)} \sim \epsilon$.

For instance in the $m > 1$ case, it follows that

$$|x| \sim \left(\frac{1}{\epsilon}e^{(1+2\beta s(m-1))f'(0)t}\right)^{1/(N+2s)} \sim e^{\frac{1+2\beta s(m-1)}{N+2s}f'(0)t},$$

and we deduce an exponential behaviour of the level sets $|x| \sim e^{\sigma_3 t}$, where $\sigma_3 = \dfrac{1+2\beta s(m-1)}{N+2s}f'(0)$. Similarly, in the $m_1 < m < 1$ case, we get that

$$|x| \sim \left(\frac{1}{\epsilon}e^{f'(0)t}\right)^{1/(N+2s)} \sim e^{\sigma_2 t}, \quad \sigma_2 = \frac{f'(0)}{N+2s}.$$

3.3 Evolution of level sets II. Convergence to 1 on inner sets

In this section, we will prove the convergence to 1 of the solution $u(x,t)$ of Problem (KPP), i. e., the second part of the statements of our main theorems 3.1.1, 3.1.2, and 3.1.3.

3.3.1 Case $m > m_1$

We will present this case in full detail. The proof for the case $m_c < m < m_1$ being similar, we will sketch it at the end of this section. We have $N \geq 1$, $s \in (0,1)$, $m > m_1$, f satisfies (3.1), and $\sigma_2 = \frac{f'(0)}{N+2s}$, as defined in (3.5).

Proposition 3.3.1. *Let $N \geq 1$, $s \in (0,1)$, $m_1 < m$, f satisfying* (3.1)*. Let u be a solution of Problem* (KPP) *with initial datum $0 \leq u_0(\cdot) \leq 1$, $u_0 \not\equiv 0$. Then for every $\sigma \in (0, \sigma_2)$, $u(x,t) \to 1$ uniformly on $\{|x| \leq e^{\sigma t}\}$ as $t \to \infty$.*

Proof. We fix $\sigma \in (0, \sigma_2)$. Proving the convergence of $u(x,t)$ to 1 is equivalent to proving the convergence of $1 - u^m$ to 0. Therefore, we fix $\lambda > 0$ and we need to find a time t_λ large enough such that $1 - u^m(x,t) \leq \lambda$ for all $t \geq t_\lambda$ and $|x| \leq e^{\sigma t}$.

• Let us accept for the moment the following lower estimate that will be proved later as Lemma 3.3.4: given $\nu \in (\sigma, \sigma_2)$, there exist $\epsilon \in (0,1)$ and $t_0 > 0$ such that

$$u \geq \epsilon, \quad \text{for } t \geq t_0 \text{ and } |x| \leq e^{\nu t}. \tag{3.16}$$

We now proceed with the last part of the argument, where the effect of the nonlinear diffusion is most clearly noticed. We take $t_1 \geq t_0$ and consider the inner sets where

$$\epsilon \leq u \leq 1 \quad \text{for } (x,t) \in \Omega_I := \{t \geq t_1,\ |x| \leq e^{\nu t}\}.$$

Let $v = 1 - u^m$. Then v satisfies the equation

$$\frac{1}{m}(1-v)^{\frac{1}{m}-1}v_t + (-\Delta)^s v + f(u) = 0, \tag{3.17}$$

that we write in the form

$$a(x,t)v_t + (-\Delta)^s v + b(x,t)v = 0, \qquad a(x,t) = \frac{1}{m}u^{1-m}, \quad b(x,t) = \frac{f(u)}{v}. \tag{3.18}$$

Moreover, we estimate $a(x,t)$ as follows

$$a_0 = \frac{1}{m}\epsilon^{1-m} \le a(x,t) \le a_1 := \frac{1}{m} \text{ in } \Omega_I, \quad \text{if } \ m < 1,$$

respectively,

$$a_0 = \frac{1}{m} \le a(x,t) \le a_1 := \frac{1}{m}\epsilon^{1-m} \text{ in } \Omega_I, \quad \text{if } m > 1.$$

We argue similarly for $b(x,t)$ in Ω_I:

$$b(x,t) = \frac{f(u)}{1-u^m} = \frac{f(u)}{(1-u)m\,\xi^{m-1}} \ge b_0, \quad \xi \in (u,1),$$

where

$$b_0 = \frac{1}{m}\frac{f(\epsilon)}{1-\epsilon}\epsilon^{1-m} \text{ if } m < 1 \quad \text{and} \quad b_0 = \frac{1}{m}\frac{f(\epsilon)}{1-\epsilon} \text{ if } m > 1.$$

In particular, v satisfies

$$a(x,t)v_t + (-\Delta)^s v + b_0 v \le 0 \quad \text{in } \Omega_I. \tag{3.19}$$

• We look for a super-solution w to Problem (3.18) that will be found as a solution to a linear problem with constant coefficients, and we also need that $w_t \le 0$. More precisely, we consider w solution of the concrete problem

$$\begin{cases} a_1 w_t(x,t) + (-\Delta)^s w(x,t) + b_0 w = 0 & \text{for } x \in \mathbb{R}^N \text{ and } t > t_1, \\ w(x,t_1) = 1 + \frac{1}{C_2}|x|^\gamma & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{3.20}$$

where the exponent γ is taken such that

$$0 < \gamma := \frac{1}{\nu}\frac{b_0}{a_1} < 2s. \tag{3.21}$$

We can eventually consider a smaller ϵ for this inequality to hold. Equation (3.20) is linear, the solution can be computed explicitly

$$w(x,t) = e^{-\frac{b_0}{a_1}(t-t_1)}\overline{w}(x,\tau), \qquad \tau = \frac{1}{a_1}(t-t_1),$$

where $\overline{w}(x,\tau)$ solves the linear problem

$$\overline{w}_\tau(x,\tau) + (-\Delta)^s \overline{w}(x,\tau) = 0, \qquad \overline{w}(0) = 1 + \frac{1}{C_2}|x|^\gamma.$$

We observe that $\overline{w}$ can be written in the following form

$$\overline{w}(x,\tau) = 1 + \frac{1}{C_2}U(x,\tau+\theta_1), \tag{3.22}$$

where

$$U(x,\tau) = \tau^{\alpha_1}F(|x|\tau^{-\beta_1}), \quad \alpha_1 = \frac{\gamma}{2s},\ \beta_1 = \frac{1}{2s},$$

is the self-similar solution of the linear problem

$$U_\tau(x,\tau) + (-\Delta)^s U(x,\tau) = 0, \qquad U(x,0) = |x|^\gamma.$$

The properties of the self-similar solutions $U(x,\tau)$ deserve a separate study, which is done in detail in Section 2.4. Thus, by Lemma 2.4.2 the profile F is non-decreasing and $U(x,\tau)$ has a spatial decay as $|x|^\gamma$ for large $|x|\tau^{-1/2s}$:

$$C_2|x|^\gamma \le U(x,\tau) \le C_1|x|^\gamma \quad \text{for all} \quad |x|\tau^{-1/2s} \ge K_1. \tag{3.23}$$

We will consider a suitable delay time τ_1 in the definition of $\overline{w}$ stated in (3.22). In what follows we will use the notation $\eta = |x|\tau^{-\beta_1}$. We check that the derivative w_t is negative:

$$\begin{aligned}
w_t(x,t) &= \frac{\partial}{\partial t}\left[e^{-\frac{b_0}{a_1}(t-t_1)}(1 + C_2^{-1}U(x,\tau+\tau_1))\right] \\
&= e^{-\frac{b_0}{a_1}(t-t_1)}\left[-\frac{b_0}{a_1}\big(1+\frac{1}{C_2}(\tau+\tau_1)^{\alpha_1}F(\eta)\big) + \frac{1}{C_2}(\tau+\tau_1)^{\alpha_1-1}\big(\alpha_1F(\eta) - \beta_1\eta F'(\eta)\big)\frac{d\tau}{dt}\right] \\
&= e^{-\frac{b_0}{a_1}(t-t_1)}\frac{1}{a_1C_2}\big[-b_0C_2 + (\tau+\tau_1)^{\alpha_1-1}\big((-b_0(\tau+\tau_1)+\alpha_1)F(\eta) - \beta_1\eta F'(\eta)\big)\big].
\end{aligned}$$

Since $F'(\eta) > 0$ for all $\eta > 0$, we get that $w_t(x,t) \le 0$ for all $t \ge t_1$ if $\tau+\tau_1 \ge \alpha_1/b_0$, which is true for a suitable choice of τ_1.

• Now we can compare w and v by applying the Maximum Principle stated in Lemma 3.4.1 of the Appendix, as in [37]. We define $\overline{v} = v - w$ and we ensure that the hypotheses of the Lemma are satisfied.

(H1) We check that $w(x,t_1) \ge v(x,t_1)$ for all $x \in \mathbb{R}^N$:

$$w(x,t_1) \ge 1 > v = 1 - u^m, \quad \forall x \in \mathbb{R}^N.$$

(H2) We check that $w \ge v$ in $\big(\mathbb{R}^N \times (t_1,\infty)\big) \setminus \Omega_I$, that is $t \ge t_1$ and $|x| \ge e^{\nu t}$. At this point, we use the estimates (3.23). We ensure that $e^{\nu t} \ge K_1(\tau+\tau_1)^{1/2s}$ for all $t \ge t_1$, which is true by choosing eventually a larger t_1. Therefore

$$\begin{aligned}
w(x,t) = e^{-\frac{b_0}{a_1}(t-t_1)}\overline{w}(\tau,x) &\ge e^{-\frac{b_0}{a_1}(t-t_1)}(1 + \frac{1}{C_2}C_2|x|^\gamma) \\
&\ge e^{-\frac{b_0}{a_1}(t-t_1)}(1 + e^{\gamma\nu t}) \ge 1 \ge v(x,t) \quad \text{for all } t \ge t_1,\ |x| \ge e^{\nu t}
\end{aligned}$$

since γ satisfies (3.21). By the previous computation, $\overline{v} \le 0$ in $\big(\mathbb{R}^N \times (t_1,\infty)\big) \setminus \Omega_I$.

(H3) Next step is to prove that $\overline{v}$ is a sub-solution of Problem (3.19). Indeed, we have that

$$
\begin{aligned}
&a(x,t)\overline{v}_t + (-\Delta)^s\overline{v} + b_0\overline{v} = \\
&a(x,t)v_t + (-\Delta)^s v + b_0 v - [a_1 w_t + (-\Delta)^s w + b_0 w] + (a_1 - a(x,t))w_t \le 0 \text{ in } \Omega_I.
\end{aligned}
$$

By Lemma 3.4.1 we obtain that $\overline{v} \le 0$ in $\mathbb{R}^N \times [t_1, \infty)$ for t_1 taken to be large enough. Thus,

$$
v(x,t) \le w(x,t) = e^{-\frac{b_0}{a_1}(t-t_1)}(1 + C_2^{-1}U(x,\tau+\tau_1)) \le e^{-\frac{b_0}{a_1}(t-t_1)}(1 + \frac{C_1}{C_2}|x|^\gamma).
$$

• Let us consider the inner set $(x,t) \in \{t \ge t_\lambda,\ |x| \le C_\lambda e^{\nu t}\}$. We have

$$
v(x,t) \le e^{-\frac{b_0}{a_1}(t-t_1)}(1 + \frac{C_1}{C_2}C_\lambda^\gamma e^{\gamma\nu t}) \le e^{-\frac{b_0}{a_1}(t_\lambda - t_1)} + \frac{C_1}{C_2}e^{\frac{b_0}{a_1}t_1}C_\lambda^\gamma \le \lambda
$$

for C_λ small enough and t_λ large enough.

Finally, since $\sigma < \nu$ then $e^{\sigma t} \le C_\lambda e^{\nu t}$ for every $t \ge t_\lambda$ with t_λ large enough, and the previous inequality implies that

$$
1 - u^m(x,t) = v(x,t) \le \lambda \qquad \text{for} \quad t \ge t_\lambda,\ |x| \le e^{\sigma t},
$$

which concludes the proof of the uniform convergence to the level $u = 1$. □

To complete the proof of the result of this subsection, we need to supply the proof of the lower estimate (3.16). This will be done in three steps.

Step I. Starting with arbitrary initial datum $0 \le u_0 \le 1$, $u_0 \ne 0$, we obtain a lower bound for u with the desired tail $u \ge c\,|x|^{-(N+2s)}$ for large $|x|$. The result corresponds to Lemma 3.3.1.

Step II. We prove that given an initial data taking the value ϵ in the ball of radius ρ_0 and decaying like that $|x|^{-(N+2s)}$ for large $|x|$, the corresponding solution of Problem (KPP) will be raised to at least the same level ϵ in a larger ball ρ_1 and in a later time that is estimated. The sizes are important. This will be Lemma 3.3.2.

Step III. By combining the previous two results, we conclude that $u \ge \epsilon$ on the inner sets, for a certain $\epsilon > 0$. This will be Lemma 3.3.3 and Lemma 3.3.4.

Steps II and III follow the ideas of [37] in the linear case, with a long technical adaptation to nonlinear diffusion.

Lemma 3.3.1 (Long Tail Behaviour). *Let $N \ge 1$, $s \in (0,1)$, $m > m_1$, f satisfying* (3.1) *and $\sigma \in (0,\sigma_2)$. Let u be the solution of Problem* (KPP) *with initial datum $u(\cdot,0) = u_0$, where $0 \le u_0 \le 1$, $u_0 \ne 0$. Then for any fixed $t_0 > 0$ there exist $\epsilon \in (0,1)$,*

$a_0 > 0$, $\rho_0 > 1$ *such that*

$$u(x,t) \geq v_0(x) := \begin{cases} a_0|x|^{-(N+2s)}, & |x| \geq \rho_0, \\ \epsilon = a_0\rho_0^{-(N+2s)}, & |x| \leq \rho_0, \end{cases}$$

for all $t \in [t_0, 2t_0]$, $x \in \mathbb{R}^N$.

Proof. We recall that $\sigma_2 = f'(0)/(N+2s)$. The idea is to view $u(x,t)$ the solution of Problem (KPP) as a super-solution of the homogeneous problem with the same initial datum u_0, that is the (FPME). Therefore,

$$u(x, t_0 + t) \geq \underline{u}(x,t), \quad \forall t \geq 0, \quad x \in \mathbb{R}^N,$$

where $\underline{u}$ is the solution of the (FPME) with initial datum u_0

$$\begin{cases} \underline{u}_t(x,t) + (-\Delta)^s \underline{u}^m(x,t) = 0 & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ \underline{u}(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{3.24}$$

We will estimate $\underline{u}$ from below by using the local and global estimates on the (FPME) given in Theorem 2.2.1 and Theorem 2.2.2 for $m < 1$, respectively Theorem 2.3.1 for $m > 1$. The decay in case $m = 1$ is well known, see Section 2.4 for a review. More exactly, in all cases $m > m_1$, there exist a time $T > 0$ and constant $R > 0$ such that

$$u(x,t) \geq C(t)|x|^{-(N+2s)}, \quad \forall |x| \geq R, \ 0 < t < T.$$

Then, for a fixed $t_* \in (0,T)$ which also satisfies $t_* < t_0$, we can find a Barenblatt solution $B_M(x,t)$ and a time $t_2 > 0$ such that

$$u(x, t_*) \geq B_M(x, t_2), \quad \forall x \in \mathbb{R}^N,$$

and therefore, by the Comparison Principle

$$u(x, t + t_*) \geq B_M(x, t + t_2), \quad \forall x \in \mathbb{R}^N, \ t \geq 0.$$

In particular, we can choose $\epsilon > 0$ such that

$$u(x,t) \geq v_0(x) := \begin{cases} a_0|x|^{-(N+2s)}, & |x| \geq \rho_0, \\ \epsilon = a_0\rho_0^{-(N+2s)}, & |x| \leq \rho_0, \end{cases}$$

for all $x \in \mathbb{R}^N, \ t \in [t_0, 2t_0]$.

$\square$

Lemma 3.3.2 (Positivity for a sequence of times). *Let* $m > m_1$*. For every* $\sigma < \sigma_2$ *there exist* $t_0 \geq 1$ *and* $0 < \epsilon_0 < 1$ *depending only on* N, s, f *and* σ *for which the following* *holds: given* $\rho_0 \geq 1$ *and* $0 < \epsilon \leq \epsilon_0$, *let* $a_0 > 0$ *be defined by* $a_0 \rho_0^{-(N+2s)} = \epsilon$; *if we take*

$$v_0(x) = \begin{cases} a_0 |x|^{-(N+2s)}, & |x| \geq \rho_0, \\ \epsilon = a_0 \rho_0^{-(N+2s)}, & |x| \leq \rho_0\,, \end{cases} \tag{3.25}$$

then the solution v *of Problem* (KPP) *with initial condition* v_0 *satisfies*

$$v(x, kt_0) \geq \epsilon \quad \textit{for } |x| \leq \rho_0 e^{\sigma k t_0}, \tag{3.26}$$

for all $k \in \{0, 1, 2, 3, ...\}$.

Proof of Lemma 3.3.2 in the case $m > 1$.

I. *Preliminary choices.* From the beginning we fix $\sigma \in (0, \sigma_2)$. We take $\delta \in (0,1)$ small enough such that

$$\frac{f(\delta)}{(N+2s)\delta} \geq \sigma, \quad \frac{f(\delta)}{(N+2s)\delta} \geq N(m-1)\beta\sigma_2. \tag{3.27}$$

For example, take δ such that

$$\frac{f(\delta)}{(N+2s)\delta} = \frac{1}{2}\left(\sigma_2 + \max\{\sigma, N(m-1)\beta\sigma_2\}\right).$$

This choice will be explained later. Next we take t_0 sufficiently large depending only on N, s, u_0 and σ such that

$$e^{t_0 2\beta s \frac{f(\delta)}{\delta}} \geq (1 + t_0/c_2)^{N\beta} K_3, \qquad (K_2/2K_1)^{1/(N+2s)} e^{\frac{f(\delta)}{(N+2s)\delta} t_0} \geq e^{\sigma t_0}, \tag{3.28}$$

where $K_2 < 2K_1$ are constants describing the properties of the profile F_1 of the Barenblatt function with mass 1 given in (2.9) and (2.10). We recall for convenience that

$$K_2 (1 + r^{N+2s})^{-1} \leq F_1(r) \leq K_1 r^{-(N+2s)}, \qquad \forall r > 0.$$

Throughout the proof there will appear several expressions involving the three parameters K_1, K_2 and $F_1(0)$. We introduce here the notations used:

$$c_1 = K_1^{-\frac{N}{N+2s}} F_1(0)^{-\frac{2s}{N+2s}}, \quad c_2 = K_1^{-\frac{2s}{N+2s}} F_1(0)^{m-1+\frac{2s}{N+2s}}, \quad K_3 = 2F_1(0) K_2^{-1}. \tag{3.29}$$

Define now ϵ_0 by

$$\epsilon_0 = \delta\; e^{-(f(\delta)/\delta)\; t_0}. \tag{3.30}$$

Now, we fix $0 < \epsilon < \epsilon_0$ and $\rho_0 > 1$.

II. *First step of the iteration* $k = 1$. We will do a very detailed analysis of the case $k = 1$, which is then iterated for the rest of values of k.

(IIa). *Construction of sub-solutions to Problem* (KPP). Let w be a solution of the problem with linearized reaction

$$\begin{cases} w_t(x,t) + (-\Delta)^s w^m(x,t) = \frac{f(\delta)}{\delta} w & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ w(0,x) = v_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{3.31}$$

We define $\overline{w}(x,\tau)$ by

$$\overline{w}(x,\tau) = e^{-\frac{f(\delta)}{\delta}t} w(x,t),$$

with a new time

$$\tau = \frac{1}{(m-1)f(\delta)/\delta}\left[e^{(m-1)\frac{f(\delta)}{\delta}t} - 1\right] \text{ if } m > 1, \tag{3.32}$$

so that $\tau = t$ in the limit $m = 1$. Then, $\overline{w}$ is a solution of the Fractional Porous Medium Equation with initial datum v_0

$$\begin{cases} \overline{w}_\tau(x,\tau) + (-\Delta)^s \overline{w}^m(x,\tau) = 0 & \text{for } x \in \mathbb{R}^N \text{ and } \tau > 0, \\ \overline{w}(x,0) = v_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{3.33}$$

(IIb). *Comparison with a Barenblatt solution. Lower bound for* $v(x,t_0)$. We prove that there exist $M_1 > 0$ and $\theta_1 > 0$ such that

$$v_0(x) \geq B_{M_1}(x,\theta_1), \quad \forall x \in \mathbb{R}^N, \tag{3.34}$$

where $B_{M_1}(x,\tau)$ is the Barenblatt solution of Problem (FPME) with mass M_1 given by Theorem 2.1.1:

$$B_{M_1}(x,\tau) = M_1 B_1\left(x, M_1^{m-1}\tau\right). \tag{3.35}$$

Now, $B_{M_1}(x,\tau)$ can be written in terms of the profile F_1 as

$$B_{M_1}(x,\tau) = M_1^{1-(m-1)\alpha}\tau^{-\alpha} F_1\left(\left(M_1^{m-1}\tau\right)^{-\beta}|x|\right). \tag{3.36}$$

We will use the properties of the profile F_1 stated in (2.9) and (2.10). With this information, we will find the constants $M_1 > 0$ and $\theta_1 > 0$ such that inequality (3.34) at the initial time holds true. For $|x| \leq \rho_0$ we have that $B_{M_1}(x,\theta_1) \leq M_1^{1-(m-1)\alpha}\theta_1^{-\alpha}F_1(0)$. Note that $1-(m-1)\alpha = 2\beta s > 0$. We impose the first condition

$$M_1^{2\beta s}\theta_1^{-\alpha}F_1(0) \leq \epsilon. \tag{3.37}$$

Next we look at the tail $|x| \geq \rho_0$. Since we have

$$B_{M_1}(x,\theta_1) \leq M_1^{2\beta s}\theta_1^{-\alpha}K_1\left(\left(M_1^{m-1}\theta_1\right)^{-\beta}, |x|\right)^{-(N+2s)}$$

in order to use this inequality for large $|x|$ we also impose the condition

$$K_1 M_1^{1+2\beta s(m-1)} \theta_1^{2\beta s} \le a_0, \quad \text{where } a_0 = \epsilon \rho_0^{N+2s}. \tag{3.38}$$

Conditions (3.37) and (3.38) are sufficient for inequality (3.34) to hold. Then, by the Comparison Principle we get

$$B_{M_1}(x, \tau + \theta_1) \le \overline{w}(x, \tau) \quad \text{for all } |x| \in \mathbb{R}^N,\ \tau > 0. \tag{3.39}$$

Putting equality in the inequalities (3.37) and (3.38) we get

$$M_1 = c_1 \epsilon \rho_0^N, \qquad \theta_1 = c_2 \epsilon^{1-m} \rho_0^{2s}, \tag{3.40}$$

(with c_1, c_2 positive constants not depending on ϵ or ρ_0). We can easily see that the expressions are dimensionally correct. The constants c_1 and c_2 were defined in (3.29). In particular, $(M_1^{m-1}\theta_1)^\beta = c_3 \rho_0$, with $c_3 = (F_1(0)/K_1)^{-1/(N+2s)}$.

Since $v_0 \le \epsilon$ in $\mathbb{R}^N$ then $\overline{w}(x,\tau) \le \epsilon$ for all $x \in \mathbb{R}^N$, $\tau > 0$, and then in terms of $w(x,t)$ we obtain the following bound

$$0 \le w(x,t) \le e^{\frac{f(\delta)}{\delta} t_0} \epsilon \le e^{f'(0) t_0} \epsilon_0 = \delta, \quad \forall t \le t_0.$$

Since $\frac{f(\delta)}{\delta}\xi \le f(\xi)$ for $0 \le \xi \le \delta$, then w is a sub-solution of Problem (KPP) in $\mathbb{R}^N \times [0, t_0]$. By the Comparison Principle and estimate (3.39), we obtain that at the moment t_0

$$v(\cdot, t_0) \ge w(\cdot, t_0) = e^{\frac{f(\delta)}{\delta} t_0} \overline{w}(\cdot, \tau_0) \ge e^{\frac{f(\delta)}{\delta} t_0} B_{M_1}(\cdot, \tau_0 + \theta_1) \ \text{ in } \mathbb{R}^N, \tag{3.41}$$

where we use the notation $\tau_0 = \tau(t_0)$ defined by (3.32).

(IIc). We will now prove that estimate (3.41) with the choices (3.40) for M_1 and θ_1 implies the lower bound (3.26) stated in Lemma 3.3.2 in the case $k = 1$, $m > 1$. Indeed, we have

$$\begin{aligned} v(x, t_0) &\ge e^{\frac{f(\delta)}{\delta} t_0} B_{M_1}(x, \tau_0 + \theta_1) \\ &= e^{\frac{f(\delta)}{\delta} t_0} M_1^{2\beta s} (\tau_0 + \theta_1)^{-\alpha} F_1\left(M_1^{-(m-1)\beta} (\tau_0 + \theta_1)^{-\beta} |x|\right) \\ &\ge e^{\frac{f(\delta)}{\delta} t_0} M_1^{2\beta s} (\tau_0 + \theta_1)^{-\alpha} K_2 \left(1 + \left(M_1^{-(m-1)\beta} (\tau_0 + \theta_1)^{-\beta} |x|\right)^{(N+2s)}\right)^{-1}. \end{aligned} \tag{3.42}$$

Our aim now is to be able to continue this estimate until we reach a bound $v_1(x)$ of the form (3.25) for the same ϵ and a larger radius ρ_1. We will choose some ρ_1 and then check that the lower bound for $v(x, t_0)$ is larger than ϵ at $|x| = \rho_1$. In order to simplify the estimate of the last parenthesis in formula (3.42), we will impose the condition

$$M_1^{-(m-1)\beta} (\tau_0 + \theta_1)^{-\beta} \rho_1 \ge 1 \tag{3.43}$$

which is natural since the radius ρ_1 in the iteration process will increase. Then we only need to have

$$v(x,t_0) \geq (K_2/2)e^{\frac{f(\delta)}{\delta}t_0} M_1^{1+2(m-1)\beta s}(\tau_0+\theta_1)^{2\beta s}\rho_1^{-(N+2s)} \geq \epsilon \quad \text{for } |x| = \rho_1. \qquad (3.44)$$

Notice that $M_1^{-(m-1)\beta}(\tau_0+\theta_1)^{-\beta}\rho_1 = c_3^{-1}\left(1+(\tau_0/\theta_1)\right)^{-\beta}(\rho_1/\rho_0)$. Hence the first condition (3.43) is equivalent to

$$\rho_1/\rho_0 \geq c_3\left(1+(\tau_0/\theta_1)\right)^{\beta},$$

while, taking into account that $M_1^{1+2(m-1)\beta s}\theta_1^{2\beta s} = a_0/K_1$ and $a_0 = \epsilon\rho_0^{N+2s}$, the second, (3.44), means that

$$(\rho_1/\rho_0)^{(N+2s)} \leq (K_2/2K_1)e^{\frac{f(\delta)}{\delta}t_0}\left(1+(\tau_0/\theta_1)\right)^{2\beta s}. \qquad (3.45)$$

Both conditions are compatible if and only if

$$e^{\frac{f(\delta)}{\delta}t_0}\left(1+(\tau_0/\theta_1)\right)^{-N\beta} \geq K_3, \qquad (3.46)$$

where $K_3 := 2K_1K_2^{-1}c_3^{N+2s} = 2F_1(0)/K_2$. Now recall that θ_1 depends on ρ_0 by (3.40), $\rho_0 \geq 1$ and θ_1 is bounded below by $\tau_* = \epsilon^{1-m}c_2$, the value for $\rho_0 = 1$. Since $m \geq 1$, $\epsilon < 1$ then $\theta_1 \geq \tau_* \geq c_2$. We see this condition as a way of choosing t_0. Let us find a simpler condition for t_0 such that the required inequality (3.46) holds true. To this aim, observe that $\tau_0 = \tau(t_0) \leq t_0\, e^{(m-1)(f(\delta)/\delta)\,t_0}$ which can be seen by definition (3.32). Then, it will be enough for t_0 to satisfy

$$e^{\frac{f(\delta)}{\delta}t_0} \geq \left(1+\frac{t_0}{c_2}e^{(m-1)(f(\delta)/\delta)t_0}\right)^{N\beta} K_3.$$

This inequality is possible when the exponents are ordered, i. e., if $(m-1)N\beta < 1$, which is true since $1-(m-1)N\beta = 2\beta s$. In particular, we can take t_0 large enough such that

$$e^{2\beta s\frac{f(\delta)}{\delta}t_0} \geq \left(1+\frac{t_0}{c_2}\right)^{N\beta} K_3.$$

This last choice of t_0 is sufficient for condition (3.46) to hold true. It is independent of ρ_0 and ϵ, and this will be used below.

Once this is guaranteed, we choose the largest possible ρ_1 satisfying (3.45), which is

$$\frac{\rho_1}{\rho_0} = (K_2/2K_1)^{1/(N+2s)}\, e^{\frac{f(\delta)}{(N+2s)\delta}t_0}\left(1+(\tau_0/\theta_1)\right)^{2s\beta/(N+2s)} := L_0. \qquad (3.47)$$

Hence formula (3.44) takes place with equality in the second inequality.

Comments on the new radius ρ_1. Notice that $\rho_1 \geq e^{\sigma t_0}\rho_0$, since t_0 satisfies (3.28).

(IId). With this choice of ρ_1 and t_0, estimate (3.44) holds. In conclusion, we have

$$v(x,t_0) \geq e^{\frac{f(\delta)}{\delta}t_0} B_{M_1}(x,\tau(t_0)+\theta_1) \geq \epsilon \quad \text{for } |x| = \rho_1 \,,$$

and thus, since the profile F_1 is non-increasing, we get that

$$v(x,t_0) \geq \epsilon, \quad \forall |x| \leq \rho_1 \,.$$

The behaviour for large $|x|$ is as follows. For $|x| \geq \rho_1$ we have $M_1^{-(m-1)}(\tau_0+\theta_1)^{-\beta}|x| \geq 1$, and, according to (3.44) and (3.47), we get that

$$v(x,t_0) \geq \epsilon \rho_1^{N+2s}|x|^{-(N+2s)}, \quad \forall |x| \geq \rho_1.$$

Remark that $\rho_0 e^{\sigma t_0} \leq \rho_1$. Finally, we define $a_1 := \epsilon\rho_1^{N+2s}$ and thus $v(\cdot,t_0) \geq v_1(\cdot)$ where v_1 is given by the expression

$$v_1(x) = \begin{cases} a_1|x|^{-(N+2s)}, & |x| \geq \rho_1; \\ \epsilon = a_1\rho_1^{-(N+2s)}, & |x| \leq \rho_1. \end{cases}$$

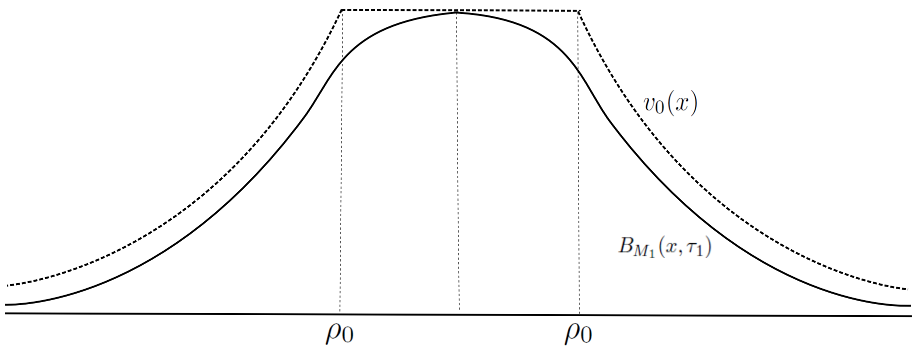


FIGURE 3.2: Step (*IIb*)

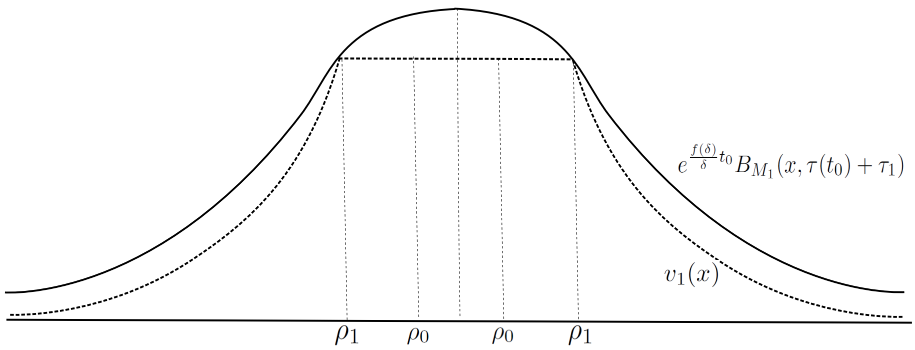


FIGURE 3.3: Step (*IIc*)

The proof is complete for $m > 1$ and $k = 1$ (see Figures 3.2 and 3.3 for the construction of v_1).

III. *The iteration.* We are now ready to address the next delicate step. Once we have proved that $v(x,t_0) \geq v_1(x)$ for all $x \in \mathbb{R}^N$, where v_1 is defined above, we apply the same proof and result to obtain

$$v(x, 2t_0) \geq (\text{solution of KPP with initial data } v_1(x))(t_0) \geq v_2(x),$$

where $v_2(x)$ has the same construction as $v_0(x)$ and $v_1(x)$ but with parameters ρ_2 and a_2. Since $\rho_1 > \rho_0$, the previous choice of t_0 is still valid to get a similar conclusion. The argument continues for all $k = 3, 4, \ldots$.

Let us check more closely the quantitative part of the iteration in order to get an improvement. In the process we keep ϵ fixed but we replace ρ_0 by ρ_k, $k \geq 1$, so that the formula (3.47) becomes

$$\frac{\rho_{k+1}}{\rho_k} = L_k := (K_2/2K_1)^{1/(N+2s)}\, e^{\frac{f(\delta)}{(N+2s)\delta}t_0} \left(1 + (\tau_0/\theta_1(\rho_k))\right)^{2s\beta/(N+2s)}.$$

Now, if we are given some $\sigma < \sigma_2 = f'(0)/(N+2s)$, we define

$$L_\infty = (K_2/2K_1)^{1/(N+2s)}\, e^{\frac{f(\delta)}{(N+2s)\delta}t_0},$$

and impose that $L_\infty \geq e^{\sigma t_0}$ by changing the definition of t_0 (note that this is compatible). Then we have $L_k \geq L_\infty \geq e^{\sigma t_0}$ for every k, so that as $k \to \infty$ we have $\rho_k \to \infty$ in an exponential way. The conditions we put on δ and t_0 can be summarized in (3.27) and (3.28), and they are independent of the parameters θ_k, ρ_k of the iteration. This ends the proof for $m > 1$.

□

Proof of Lemma 3.3.2 in the case $m < 1$. The outline of the proof is similar to the case $m > 1$. We explain the differences that appear, that are not technically trivial.

I. *Preliminary choices.* From the beginning we fix $\sigma \in (0, \sigma_2)$ and $\rho_0 >> 1$. We take $\delta \in (0, 1)$ small enough such that

$$\frac{f(\delta)}{(N+2s)\delta} > \sigma.$$

We take t_0 large enough such that

$$\frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} \geq F_1(0)2^{\beta N}, \qquad (K_2/2K_1)^{1/(N+2s)}\, e^{\frac{f(\delta)}{(N+2s)\delta}t_0} \geq e^{\sigma t_0}.$$

Notice that (i) δ depends only on σ; (ii) t_0 depends only on σ, δ and some constants appearing in the characterization of the Barenblatt function.

In this case we introduce the new time τ via

$$\tau = \frac{1}{(1-m)f(\delta)/\delta}\left[1 - e^{-(1-m)\frac{f(\delta)}{\delta}t}\right] \text{ if } m < 1. \tag{3.48}$$

Therefore, for each t we have a new bounded time $\tau(t) \leq \tau_\infty = 1/((1-m)f(\delta)/\delta)$. For $t = t_0$ we denote the corresponding $\tau(t_0) =: \tau_0$.

Next, we define ϵ_0 by $e^{(f(\delta)/\delta)t_0}\epsilon_0 = \delta$. Fix $0 < \epsilon < \epsilon_0$. At this moment, we ensure that the first radius ρ_0 appearing in the proof is large enough, more exactly, we ask ρ_0 to satisfy

$$\frac{1}{(1-m)f(\delta)/\delta}\ \epsilon^{-(1-m)} \le c_2\ \rho_0^{2s} \tag{3.49}$$

This condition says that $\rho_0 = \rho_0(\epsilon)$ is sufficiently large depending on ϵ.

These values are set before starting the proof of the first step $k = 1$. Hence, these values will be the same during the iteration process.

II. *First step* $k = 1$. We consider the initial data v_0 defined by (3.25). We take M_1 and θ_1 satisfying conditions (3.37) and (3.38):

$$M_1^{2\beta s}\theta_1^{-\alpha}F_1(0) = \epsilon, \quad K_1 M_1^{1+2\beta s(m-1)}\theta_1^{2\beta s} = \epsilon\rho_0^{N+2s}.$$

Therefore $M_1 = c_1\epsilon\rho_0^N$ and $\theta_1 = c_2\epsilon^{1-m}\rho_0^{2s}$. In what follows, we will need ρ_0 large enough such that

$$\tau_0 \le \theta_1. \tag{3.50}$$

Hence, it is sufficient to have $\tau_\infty \le \theta_1$, which is satisfied for $\rho_0 = \rho_0(\epsilon)$ large enough according to the previous choice (3.49).

Then, at point (IIc) of the previous proof we have

$$\begin{aligned} v(x,t_0) &\ge e^{\frac{f(\delta)}{\delta}t_0} M_1^{2\beta s}(\tau_0+\theta_1)^{-\alpha}F_1\left(M_1^{-(m-1)\beta}(\tau_0+\theta_1)^{-\beta}|x|\right) \\ &\ge e^{\frac{f(\delta)}{\delta}t_0} M_1^{2\beta s}(\tau_0+\theta_1)^{-\alpha}K_2\left(1+(M_1^{-(m-1)\beta}(\tau_0+\theta_1)^{-\beta}|x|)^{(N+2s)}\right)^{-1}. \end{aligned}$$

Our purpose is to find suitable ρ_1 such that $v(x,t_0) \ge v_1(x)$ for all $x \in \mathbb{R}^N$, where $v_1(x)$ is defined as

$$v_1(x) = \begin{cases} a_1|x|^{-(N+2s)}, & |x| \ge \rho_1, \\ \epsilon = a_1\rho_1^{-(N+2s)}, & |x| \le \rho_1. \end{cases} \tag{3.51}$$

Since the profile F_1 is non-increasing, the idea to find ρ_1 such that when $|x| = \rho_1$

$$\begin{aligned} v(x,t_0) &\ge e^{\frac{f(\delta)}{\delta}t_0} M_1^{2\beta s}(\tau_0+\theta_1)^{-\alpha}K_2\left(1+\left(M_1^{-(m-1)\beta}(\tau_0+\theta_1)^{-\beta}\rho_1\right)^{(N+2s)}\right)^{-1} \\ &\ge a_1\rho_1^{-(N+2s)} = \epsilon. \end{aligned}$$

In particular, if we take

$$M_1^{-(m-1)\beta}(\tau_0+\theta_1)^{-\beta}, \rho_1 \ge 1 \tag{3.52}$$

then, for $|x| = \rho_1$,

$$
\begin{aligned}
v(x,t_0) &\geq e^{\frac{f(\delta)}{\delta}t_0} M_1^{2\beta s}(\tau_0+\theta_1)^{-\alpha} K_2 \left(2\left(M_1^{-(m-1)\beta}(\tau_0+\theta_1)^{-\beta}\rho_1\right)^{(N+2s)}\right)^{-1} \\
&= \frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} M_1^{2\beta s+(m-1)\beta(N+2s)}(\tau_0+\theta_1)^{-\alpha+\beta(N+2s)}\rho_1^{-(N+2s)} \\
&= \frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} M_1^{1+2\beta s(m-1)}(\tau_0+\theta_1)^{2\beta s}\rho_1^{-(N+2s)}.
\end{aligned}
$$

We take ρ_1 the largest value satisfying $(K_2/2)e^{\frac{f(\delta)}{\delta}t_0} M_1^{1+2\beta s(m-1)}(\tau_0+\theta_1)^{2\beta s}\rho_1^{-(N+2s)} \geq \epsilon$, that is

$$
\frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} M_1^{1+2\beta s(m-1)}(\tau_0+\theta_1)^{2\beta s}\rho_1^{-(N+2s)} = \epsilon. \tag{3.53}
$$

Then for $|x| = \rho_1$ we have

$$
v(x,t_0) \geq \epsilon = a_1\rho_1^{-(N+2s)}, \quad a_1 := \frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} M_1^{1+2\beta s(m-1)}(\tau_0+\theta_1)^{2\beta s}.
$$

For $|x| \leq \rho_1$ we have $v(x,t_0) \geq \epsilon$ and for $|x| \geq \rho_1$ we have $v(x,t_0) \geq a_1|x|^{-(N+2s)}$ justified as in the previous case $m > 1$.

It remains to check that conditions (3.52) and (3.53) are compatible, that is we need t_0 such that

$$
\frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} M_1^{1+2\beta s(m-1)}(\tau_0+\theta_1)^{2\beta s}\epsilon^{-1} = \rho_1^{N+2s} > \left((M_1^{(m-1)\beta}(\tau_0+\theta_1)^{\beta}\right)^{N+2s}.
$$

This is equivalent to

$$
\frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} M_1^{2\beta s}\theta_1^{-\beta N}\epsilon^{-1} \geq \left(1+\frac{\tau_0}{\theta_1}\right)^{\beta N}.
$$

According to the definition of M_1 and θ_1 this rewrites as

$$
\frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} \geq F_1(0)\left(1+\frac{\tau_0}{\theta_1}\right)^{\beta N}.
$$

Since $\tau_0/\theta_1 \leq 1$, then a sufficient condition for t_0 would be

$$
\frac{K_2}{2} e^{\frac{f(\delta)}{\delta}t_0} \geq F_1(0)2^{\beta N}.
$$

Notice that this condition on t_0 is independent on ϵ and ρ_0. Hence t_0 is the same in the iteration, that is done as before.

Comments on the new radius ρ_1. By the definition formula (3.53) we have

$$
\left(\frac{\rho_1}{\rho_0}\right)^{N+2s} = \frac{K_2}{2K_1} e^{\frac{f(\delta)}{\delta}t_0} \left(1+\frac{\tau_0}{\theta_1}\right)^{2\beta s}.
$$

Now, if t_0 is such that $(K_2/2K_1)e^{\frac{f(\delta)}{\delta}t_0} \geq e^{(N+2s)\sigma t_0}$, then we get that

$$\rho_1/\rho_0 \geq e^{\sigma t_0}.$$

III. *The iteration.* We point out that for the next step $k = 2$, the first radius is ρ_1. The value of ρ_1 was defined in (3.53) and satisfies $\rho_1 > \rho_0$. Hence, ρ_1 satisfies the preliminary condition (3.49) and is a good candidate for the initial radius.

The rest of the proof follows as in the case $m > 1$.

Remark. We summarize the results proved so far as follows: for small ϵ fixed, we found a ρ_0 sufficiently large (depending on ϵ) such that the line $v(x,t) = \epsilon$ propagates with exponential speed σ. When taking a smaller ϵ and a larger value of ρ_0, the proof also works. Hence, the result of exponential propagation is true for all small $\epsilon < \epsilon_0$.

□

Proof of Lemma 3.3.2 in the case $m = 1$. No change of the time variable is needed in this case, that is $\tau = t$. The proof follows similarly to the case $m > 1$. We do not give more details here, since the result for $m = 1$ has been proved in [37].

□

Lemma 3.3.3 (Expansion of uniform positivity for all times)**.** *Let $N \geq 1$, $s \in (0,1)$, $m_1 < m$, f satisfying* (3.1) *and $\sigma \in (0, \sigma_2)$. Let $t_0 > 0$ from Lemma* 3.3.2*. Then for every measurable initial datum u_0 with $0 \leq u_0 \leq 1$, $u_0 \not\equiv 0$, there exist $\epsilon \in (0,1)$ and $b > 0$ (both depending on u_0) such that the solution u of Problem* (KPP) *with initial datum $u(0,\cdot) = u_0$ satisfies*

$$u(x,t) \geq \epsilon \text{ for all } t \geq t_0 \text{ and } |x| \leq be^{\sigma t}.$$

Proof. Let t_0 defined in Lemma 3.3.2. Then, by Lemma 3.3.1, there exist $\epsilon > 0$, $a_0 > 0$, $\rho_0 > 1$ such that $u(x,t)$ is bounded from below by a function v_0 with the long tail behaviour at infinity

$$u(x,t) \geq v_0(x) := \begin{cases} a_0|x|^{-(N+2s)}, & |x| \geq \rho_0, \\ \epsilon = a_0\rho_0^{-(N+2s)}, & |x| \leq \rho_0 \end{cases}$$

for all $x \in \mathbb{R}^N$, $t \in [t_0, 2t_0]$. In this way v_0 can be taken as the initial datum (3.25) in Lemma 3.3.2. If necessary, we make a_0 smaller and ρ_0 larger to fit in the context of Lemma 3.3.2. We recall the necessary conditions: $\epsilon \leq \epsilon_0$ in (3.30) and $\rho_0 \geq \rho(\epsilon)$ in (3.49).

Therefore, by applying Lemma 3.3.2, the solution u will be raised an ϵ at a large time $\tau_0 + t_0$ and this holds true for all $\tau_0 \in [t_0, 2t_0]$. More exactly, by (3.26), for every $k = 0, 1, 2, \ldots$ one has

$$u(x, \tau_0 + kt_0) \geq \epsilon \text{ for all } |x| \leq e^{\sigma k t_0}\rho_0, \ \tau_0 \in [t_0, 2t_0]$$

which rewrites as

$$u(x,t) \ge \epsilon \text{ for all } |x| \le e^{\sigma k t_0}\rho_0, \quad t \in [(k+1)t_0, (k+2)t_0]. \tag{3.54}$$

But for $t \in [(k+1)t_0, (k+2)t_0]$ we get $e^{\sigma k t_0} = e^{\sigma k t_0 - \sigma t}e^{\sigma t} \ge e^{-2\sigma t_0}e^{\sigma t}$ and then (3.54) implies, in particular, that

$$u(x,t) \ge \epsilon, \text{ for all } |x| \le e^{-2\sigma t_0}e^{\sigma t}\rho_0, \quad t \in [(k+1)t_0, (k+2)t_0].$$

Since the union of the intervals $[(k+1)t_0, (k+2)t_0]$ with $k = 0, 1, 2, ...$ covers all $[t_0, \infty)$, we deduce that

$$u(x,t) \ge \epsilon \quad \text{if} \quad t \ge t_0 \text{ and } |x| \le \rho_0 e^{-\sigma 2 t_0}e^{\sigma t}.$$

The proof of the lemma follows by denoting $b = \rho_0 e^{-\sigma 2 t_0}$. □

Lemma 3.3.4. *Let $N \ge 1$, $s \in (0,1)$, $m > m_1$ and f satisfying* (3.1). *Let $\sigma_2 = \frac{f'(0)}{N+2s}$. Let u be a solution of Problem* (KPP) *with initial datum $0 \le u_0(\cdot) \le 1$, $u_0 \ne 0$. Then for every $\sigma < \sigma_2$ there exist $\epsilon \in (0,1)$ and $\underline{t} > 0$ such that*

$$u(x,t) \ge \epsilon \quad \textit{for all } t \ge \underline{t} \textit{ and } |x| \le e^{\sigma t}.$$

Proof. We apply Lemma 3.3.3 with σ replaced by $\sigma' \in (\sigma, \sigma_2)$. Since $e^{\sigma t} \le b e^{\sigma' t}$ for t large, where b is the constant in the statement of Lemma 3.3.3, we deduce that

$$u(x,t) \ge \epsilon \text{ for } t \ge \underline{t} \text{ and } |x| \le e^{\sigma t}.$$

□

3.3.2 Case $m_c < m < m_1$

In a similar way, we can prove the convergence to 1 on the inner sets also in the range of parameters $m_c < m < m_1$.

Proposition 3.3.2. *Let $N \ge 1$, $s \in (0,1)$, $m_c < m < m_1$, f satisfying* (3.1). *Let $\sigma_1 = \frac{1-m}{2s}f'(0)$. Let u be a solution of Problem* (KPP) *with initial datum $0 \le u_0(\cdot) \le 1$, $u_0 \ne 0$. Then for every $\sigma \in (0, \sigma_1)$, $u(x,t) \to 1$ uniformly on $\{|x| \le e^{\sigma t}\}$ as $t \to \infty$.*

Proof. We argue in a similar way as in the case $m > m_1$ proved in Proposition 3.3.1. The difference appears when obtaining the positivity on inner sets. To this aim, we start with nontrivial initial data $0 \le u_0 \le 1$ and we prove the analogue of Lemma 3.3.3. The key ingredient is to use the quantitative lower estimates for the solution $\underline{u}(x,t)$ of the Fractional Fast Diffusion Equation stated in Theorem 2.2.3 to obtain an estimate of the form

$$\underline{u}(x,t) \ge v_0(x), \quad \forall t \in [t_0, 2t_0], \ x \in \mathbb{R}^N,$$

where $v_0(x)$ is defined as

$$v_0(x) = \begin{cases} a_0|x|^{-2s/(1-m)}, & |x| \geq \rho_0, \\ \epsilon = a_0\rho_0^{-2s/(1-m)}, & |x| \leq \rho_0\,. \end{cases} \tag{3.55}$$

Afterwards, we can prove an analogue result to Lemma 3.2.1 starting with initial data of the form (3.55). Since the Barenblatt solution has a long tail decay of the form $|x|^{-2s/(1-m)}$, then we find $M_1 > 0$ and $\theta_1 > 0$ such that

$$v_0(x) \geq B_{M_1}(x, \theta_1), \quad \forall x \in \mathbb{R}^N.$$

$\square$

3.3.3 Numerical computations

The following graphics (Figures 3.4 and 3.5) have been performed using the numerical scheme developed by Felix del Teso and Juan Luis Vázquez [53].

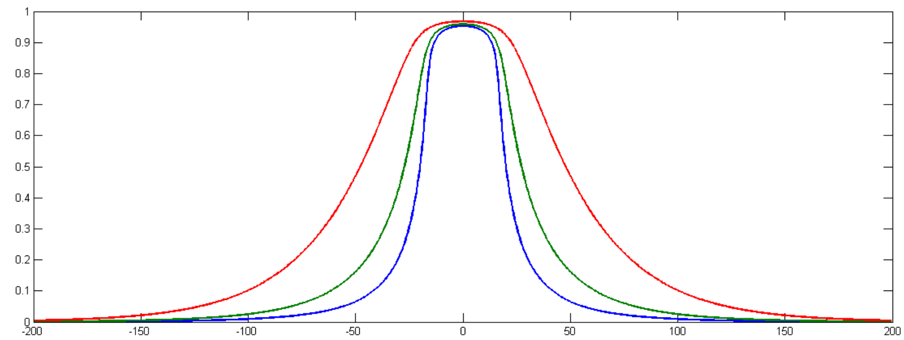


FIGURE 3.4: The solution $u(x,t)$ at different times.

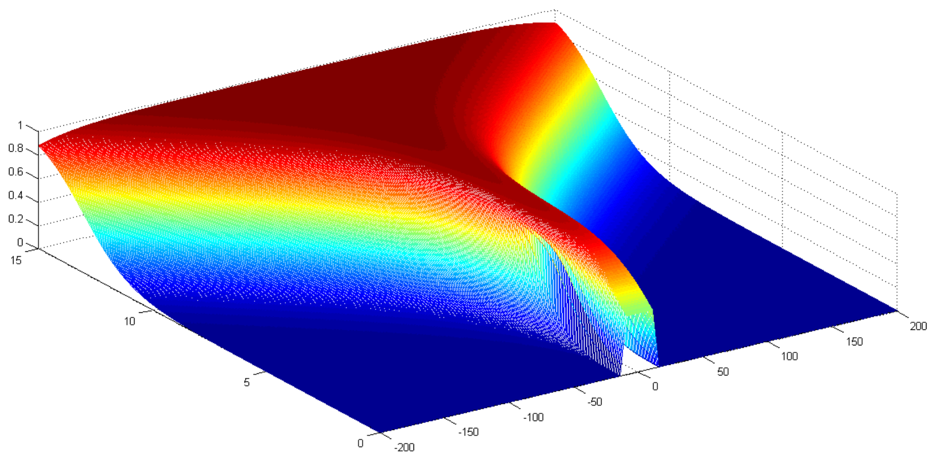


FIGURE 3.5: The solution $u(x,t)$ starting from initial data with compact support (case $m = 10$).

3.4 Appendix C: A version of the Maximum Principle

We need an interesting version of Maximum Principle proved by Cabré and Roquejoffre in [37], Lemma 2.9, suitable for comparisons in which fractional Laplacian operators are involved.

For $0 \le \gamma < 2s$ we consider functions $v : \mathbb{R}^N \to \mathbb{R}$ such that

$$|v(x)| \le C(1+|x|^\gamma) \text{ in } \mathbb{R}^N \text{ for some constant } C, \tag{3.56}$$

$$\begin{cases} \text{for every } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that:} \\ \text{if } x \in \mathbb{R}^N \text{ and } |z| \le \delta, \text{ then } |u(x+z) - u(x)| \le \epsilon(1+|x|^\gamma). \end{cases} \tag{3.57}$$

We define $X_\gamma = \{u : \mathbb{R}^N \to \mathbb{R} :\ u \text{ satisfies (3.56) and (3.57)}\}$ and D_γ is the domain of the operator $(-\Delta)^s = (-\Delta)^s$ in X_γ.

Lemma 3.4.1. *Let $N \ge 1$, $s \in (0,1)$, $0 \le \gamma < 2s$. Let $v \in C^1([0,\infty); X_\gamma)$ satisfy $v(\cdot,t) \in D_\gamma$ for all $t > 0$. Let $r : (0,\infty) \to (0,\infty)$ be a continuous function and define*

$$\Omega_I = \{(x,t) \in \mathbb{R}^N \times (0,\infty) : |x| < r(t)\}.$$

Assume in addition:

$(H1)$ $v(\cdot,0) \le 0$ *in* $\mathbb{R}^N$.

$(H2)$ $v \le 0$ *in* $(\mathbb{R}^N \times (0,\infty)) \setminus \Omega_I$.

$(H3)$ $a(x,t)v_t + (-\Delta)^s v \le b\, v$ *in* Ω_I.

Then $v \le 0$ *in* $\mathbb{R}^N \times (0,\infty)$.

Although the equation we have is different, the proof as in [37] still works (with inessential modifications).

3.5 Comments and open problems

• There are critical values of the speed σ which we do not cover in this work: σ_1 for $m_c < m < m_1$; σ_2 for $m_1 < m \le 1$; respectively, (σ_2, σ_3) for $m > 1$. The analysis of those cases leads to long new developments.

• Is there a definite profile function that represents up to translation the shape of the solution in the region where it varies in a marked way to join the level $u = 1$ to the level $u = 0$? Maybe for $s = 1/2$ this question is easier.

• For reasons of length and novelty, the case $m < m_c$ is not studied. For the corresponding fractional fast diffusion equation there appears the phenomenon of extinction in finite time. King and McCabe in [75] give an idea on the asymptotics in this range of parameters.

• A detailed numerical treatment of these problems for the case $m \ne 1$ is needed, see in this respect [52, 53].

• There are other interesting directions in this class of problems. Thus, in a recent paper [36], the authors investigate the model

$$u_t(x,t) + Au(x,t) = \mu(x)u - u^2, \quad x \in \mathbb{R}^N, \ t > 0,$$

where the function μ is supposed periodic in each spatial variable x_i and satisfy $0 < \min \mu \leq \mu(x)$.

• **Initial data with slow decay.** The decay of the initial data plays an important role in the propagation of level sets. The slower the decay is, the faster the propagation. We comment on some recent work on the issue.

In [66], Hamel and Roques consider the one dimensional classical Fisher-KPP problem $u_t = u_{xx} + f(u)$ with initial data u_0 that are assumed to decay at infinity more slowly than any exponentially decaying function. A precise quantitative estimate of the level sets of the solution is obtained in terms of the decay of the initial data, and this implies an exponential rate of propagation of level sets.

As for the Fisher-KPP with fractional diffusion, the case of slowly decay initial conditions has been recently treat by Felmer and Yangari in [59, 114] for the linear case $u_t + (-\Delta)^s u = f(u)$. Assuming the initial data satisfy $u_0(x) \geq |x|^{-b}$ with $b \in (0, 2s)$, they prove that the level sets of the solution propagate exponentially with a faster speed, thus completing the case studied by Hamel and Roques to all $s \in (0,1)$.

As far as we know, slowly decaying initial data have not been considered for nonlinear fractional diffusion cases. For our model, the proof of the convergence to 1 still works, since the pure diffusion problem, whose solutions are sub-solutions for Problem KPP, reaches a tail-type behaviour at a larger time. As for the convergence to 0 in the far field, we mention that our proof does not adapt to data with slower decay since the main technique is using the long-tail behaviour of the Barenblatt solution.

Chapter 4

The Porous Medium Equation with fractional potential pressure

We study a porous medium equation with fractional potential pressure:

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla P), \quad P = (-\Delta)^{-s} u,$$

for $m > 1$, $0 < s < 1$ and $u(x,t) \geq 0$. The problem is posed for $x \in \mathbb{R}^N$, $N \geq 1$, and $t > 0$. The initial data $u(x,0)$ is assumed to be a bounded function with compact support or fast decay at infinity. We establish existence of a class of weak solutions in the range $m \in (1,3)$ for which we determine whether the property of compact support is conserved in time depending on the parameter m, starting from the result of finite propagation known for $m = 2$. The proof of the existence result for $m \geq 3$ is still under study now and will appear in a forthcoming paper in collaboration with J.L. Vázquez and F. del Teso, [95].

Assuming the existence of a solution also for $m \geq 3$, we prove that the problem has finite speed of propagation for all $m \geq 2$, while when $m \in (1,2)$ we prove that the problem has infinite speed of propagation.

The results were announced without proofs in [94]. In this chapter we will give the detailed proofs of these results. There will be some technical lemma that we do not prove here, the proofs will appear in the forthcoming paper [95].

4.1 Introduction

1.1. Motivation and previous results. Our work is motivated by two recent problems:

(I) Porous medium equation with nonlocal diffusion effects. In [39], Caffarelli and Vázquez proposed the following model of porous medium equation with nonlocal diffusion effects

$$\partial_t u = \nabla \cdot (u \nabla P), \quad P = \mathcal{K}(u). \tag{CV}$$

The pressure P is related to u via a linear positive operator $\mathcal{K}$, which is assumed to be the inverse of the fractional Laplacian: $\mathcal{K} = (-\Delta)^{-s}$. The problem is posed for $x \in \mathbb{R}^N$, $N \geq 1$ and $t > 0$ with bounded and compactly supported initial data. The study of this model has been performed in three successive papers as follows:

(i) In [39], Caffarelli and Vázquez developed the theory of existence of bounded weak solutions that propagate with finite speed.

(ii) In [40], the same authors proved the asymptotic time behaviour of the solution. Self-similar non-negative solutions are obtained by solving an elliptic obstacle problem with fractional Laplacian for the pair pressure-density, called obstacle Barenblatt solutions.

(iii) Finally, in [38], Caffarelli, Soria and Vázquez considered the regularity and the $L^1 - L^\infty$ smoothing effect.

The study of fine asymptotic behaviour (rates of convergence) for (CV) is being studied presently by Carrillo, Huang and Vázquez [43] in the one dimensional setting.

(II) Modeling dislocation dynamics as a continuum. The equation with $s = 1/2$ in dimension $N = 1$ has been proposed by Head [67] to describe the dynamics of dislocation in crystals. The model is written in the integrated form as

$$v_t + |v_x|(-\partial^2/\partial_{xx})^{1/2} v = 0.$$

The dislocation density is $u = v_x$. This model has been recently studied by Biler, Karch and Monneau in [22], where they prove that the problem enjoys the properties of uniqueness and comparison of viscosity solutions.

1.2. Main results. In this paper we study a generalization of model (CV) to an equation with power-type nonlinearity:

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla P), \quad P = \mathcal{K}(u), \tag{4.1}$$

for $m > 1$ and $u(x,t) \geq 0$. The problem is posed for $x \in \mathbb{R}^N$, $N \geq 1$, and $t > 0$, and we give initial conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{4.2}$$

where

$$u_0 : \mathbb{R}^N \to [0,\infty) \text{ is bounded with compact support or fast decay at infinity.} \tag{A1}$$

The pressure P is related to u through a linear fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$ for $0 < s < 1$ with kernel $K(x,y) = c|x-y|^{-(N-2s)}$ (i.e. a Riesz operator). This operator is homogeneous of degree $2s$ and this fact will be important in the following proofs. When considering $m = 2$ in (4.1), we recover Problem (CV).

We first propose a definition of solution and establish the existence and main properties of the solutions.

Definition 4.1.1. *We say that u is a weak solution of* (4.1) − (4.2) *in* $Q_T = \mathbb{R}^N \times (0,T)$ *with nonnegative initial data* $u_0 \in L^1(\mathbb{R}^N)$ *if (i)* $u \in L^1(Q_T)$, *(ii)* $\mathcal{K}(u) \in L^1(0,T : W^{1,1}_{loc}(\mathbb{R}^N))$, *(iii)* $u^{m-1}\nabla\mathcal{K}(u) \in L^1(Q_T)$, *and (iv)*

$$\int_0^T \int_{\mathbb{R}^N} u\phi_t dxdt - \int_0^T \int_{\mathbb{R}^N} u^{m-1}\nabla\mathcal{K}(u)\nabla\phi dxdt + \int_{\mathbb{R}^N} u_0(x)\phi(x,0)dx = 0 \qquad (4.3)$$

holds for every test function ϕ in Q_T such that $\nabla\phi$ is continuous, ϕ has compact support in $\mathbb{R}^N$ for all $t \in (0,T)$ and vanishes near $t = T$.

The following is our most important contribution, which deals with the property of finite propagation of the solutions just constructed depending on the value of m.

Theorem 4.1.1. *Let $m \in (1,2)$ (and $s \in (0,1/2)$ if $N = 1$). Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there exists a weak solution u of equation* (4.1) *with initial data u_0. Moreover, u has the following properties:*

1. ***(Regularity)*** $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $u \in L^\infty(\mathbb{R}^N \times (0,T))$, $\nabla\mathcal{H}(u) \in L^2(\mathbb{R}^N)$.
2. ***(Conservation of mass)*** *For all $t > 0$ we have* $\displaystyle\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx$.
3. *(L^∞ **estimate**) For all $t > 0$ we have* $||u(\cdot,t)||_\infty \le ||u_0||_\infty$.
4. ***(Energy estimate)*** *For all $t > 0$ the following estimate holds*

$$C\int_0^t \int_{\mathbb{R}^N} |\nabla\mathcal{H}(u)|^2 dxdt + \int_{\mathbb{R}^N} u(t)^{3-m}dx = \int_{\mathbb{R}^N} u_0^{3-m}dx,$$

with $C = (2-m)(3-m) > 0$.

Theorem 4.1.2. *Let $m \in [2,3)$. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be such that*

$$0 \le u_0(x) \le Ae^{-a|x|} \text{ for some } A, a > 0. \qquad (4.4)$$

Then there exists a weak solution u of equation (4.1) *with initial data u_0 such that u satisfies the properties* 1, 2, 3 *of Theorem 4.1.1. Moreover, the solution decays exponentially in $|x|$ and the energy estimate holds in the form*

$$\int_0^t \int_{\mathbb{R}^N} |\nabla\mathcal{H}(u)|^2 dxdt - |C|\int_{\mathbb{R}^N} u(t)^{3-m}dx = -|C|\int_{\mathbb{R}^N} u_0^{3-m}dx$$

where $C = C(m) = \frac{1}{(2-m)(3-m)}$.

We are able to prove the exponential tail decay also in the case $m \ge 3$. However, this is not enough to obtain existence of the solution for $m \ge 3$ since we are not able to continue the same compactness arguments as in the case $m < 3$. Throughout this chapter we will state explicitly these differences.

Theorem 4.1.3. *Let $m \in (2, \infty)$. The solution to problem (4.1)-(4.2) has the property of* ***finite speed of propagation*** *in the sense that, if u_0 is compactly supported, then for any $t > 0$, $u(\cdot, t)$ is also compactly supported. On the other hand, if $N = 1$, $m \in (1, 2)$ and $s \in (0, 1/2)$, the solution u has* ***infinite speed of propagation*** *in the sense that: even if the initial data is compactly supported, for any $t > 0$ and any $R > 0$, the set $\mathcal{M}_{R,t} = \{x : |x| \geq R,\ u(x,t) > 0\}$ has positive measure.*

Remarks. (i) Finite propagation for the case $m = 2$ has been proved in [39].

(ii) A main difficulty in the work to be done is the possible lack of uniqueness and comparison of the solutions, already noticed in [39]. This is reflected in the indirect statement of the last part of the theorem.

(iii) Since $\nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u)$ tends to $-(-\Delta)^{1-s} u$ as $m \to 1$, equation (4.1) becomes $u_t + (-\Delta)^{1-s} u = 0$, known as the fractional Heat Equation, which has infinite speed of propagation. This propagation property is inherited by more general diffusion models as $u_t + (-\Delta)^s u^m = 0$, called the fractional porous medium equation, which has been studied in [46, 47, 107]. Therefore, a change in the behaviour of the solutions for some $m > 1$ was expected. We also motivate the result by numerical computations based on the scheme proposed by Teso and Vázquez in [52, 53]. We give in the graph below (Figure 4.1) a description of the result on finite propagation for different related models of nonlinear diffusion with or without fractional effects, see [20, 21, 47, 108].

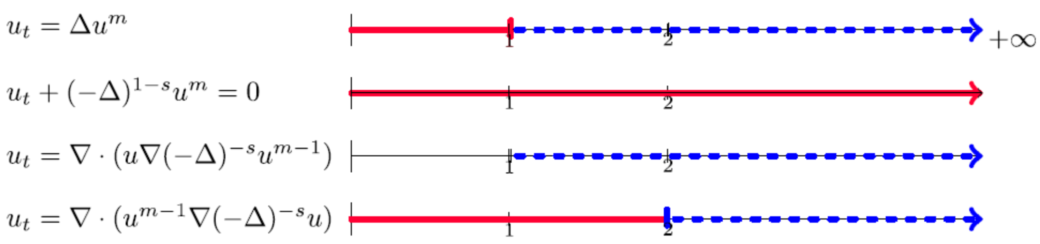


FIGURE 4.1: Ranges of exponent $m \in (0, \infty)$. Continuos red line means infinite propagation, dotted blue line means finite propagation.

1.3. Organization of the proofs

• In Section 4.2 we derive useful energy estimates. Due to the differences in the computations, we will separate the cases $m \neq 2, 3$ and $m = 3$.

• In Section 4.3 we prove the existence of a weak solution of Problem (4.1) as the limit of a sequence of solutions to suitable approximate problems.

• Section 4.5 deals with the property of finite speed of propagation for $m \geq 2$.

• In Section 4.6 we prove the infinite speed of propagation for $m \in (1, 2)$ in the one-dimensional case.

1.4. Connection with nonlinear parabolic problems of porous medium type

The classical nonlinear diffusion problem, $u_t = \Delta u^m$, the Porous Medium Equation (PME) with $m > 1$ has the property of finite speed of propagation. The situation completely changes for the PME as we take $m < 1$, i.e. the Fast Diffusion Equation, when the model has infinite speed of propagation. In both cases the behaviour of a general solution $u(x,t)$ is given by explicit self-similar solutions, also know as Barenblatt profiles [103].

Our model of nonlocal diffusion equation satisfies the property of finite speed of propagation for $m \geq 2$. This property has been proved by Caffarelli and Vázquez in the case $m = 2$ and we extend the proof here to all $m > 2$.

In the case $1 < m < 2$ the situation changes completely. The solution of Problem (4.1) has infinite speed of propagation. This result has been motivated by numerical computations.

The finite propagation property is not transmitted to other fractional diffusion models, for example $u_t = -(-\Delta)^s u^m$, with $m > (N-2s)_+/N$, the Fractional Porous Medium Equation. This model has infinite speed of propagation and the existence of fundamental solutions of self-similar type or Barenblatt solutions is known. We refer to the recent works [46, 47, 107].

Notations. We will use the notation $L_s = (-\Delta)^s$ with $0 < s < 1$ for the fractional Laplacian operator defined on smooth functions in $\mathbb{R}^N$ and extended in the natural way to the fractional Sobolev spaces $H^{2s}(\mathbb{R}^N)$. For technical reasons we will only consider the case $s < 1/2$ in one dimensional space. The inverse operator is denoted by $\mathcal{K}_s = (-\Delta)^{-s}$ and can be represented by convolution

$$\mathcal{K}_s = K_s * u, \quad K_s(x) = c(N,s)|x|^{-(N-2s)}.$$

We will also write $\mathcal{H}_s = \mathcal{K}_s^{1/2}$ which has kernel $K_{s/2}$. We will write $\mathcal{K}$ and $\mathcal{H}$ when s is fixed and known. We refer to [82] for the arguments of potential theory used throughout the paper.

For functions depending on x and t, convolution is applied for fixed t with respect to the spatial variables and we then write $u(t) = u(\cdot, t)$.

4.2 Basic estimates

In order to prove existence of weak solutions, we need a process based on several approximations that could hide to the reader the main properties of the solutions. In what follows, we perform formal computations on the solution of Problem (4.1), for which we assume smoothness, integrability and fast decay as $|x| \to \infty$. These computations will be justified later by the approximation process.

We fix $s \in (0,1)$ and $m \geq 1$. Let u be the solution of Problem (4.1) with initial data $u_0 \geq 0$. We assume $u \geq 0$ for the beginning. This property will be proved later.

• **Conservation of mass:**

$$\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_t dx = \int_{\mathbb{R}^N} \nabla \cdot (u^{m-1}\nabla\mathcal{K}(u))dx = 0. \tag{4.5}$$

• **First energy estimate:** The estimates here are significantly different depending on the exponent m. Therefore, we consider the cases:

CASE $m = 3$:

$$\frac{d}{dt}\int_{\mathbb{R}^N} \log u(x,t)dx \;=\; \int_{\mathbb{R}^N} \frac{u_t}{u}dx = \int_{\mathbb{R}^N} \nabla u\cdot\nabla\mathcal{K}(u) = \int_{\mathbb{R}^N} |\nabla\mathcal{H}(u)|^2 dx.$$

Therefore, by the conservation of mass (4.5) we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^N}(u-\log u)dx = -\int_{\mathbb{R}^N}|\nabla\mathcal{H}(u)|^2dx. \tag{4.6}$$

CASE $m \neq 3$:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} u^{3-m}(x,t)dx &= (3-m)\int_{\mathbb{R}^N} u^{2-m}u_t dx = (3-m)\int_{\mathbb{R}^N} u^{2-m}\nabla(u^{m-1}\nabla\mathcal{K}(y))dx\\ &= -(3-m)(2-m)\int_{\mathbb{R}^N}\nabla u\cdot\nabla\mathcal{K}(u)dx = -C\int_{\mathbb{R}^N}|\nabla\mathcal{H}(u)|^2dx.\end{aligned}$$

Here $C = (3-m)(2-m)$ is negative for $m \in (2,3)$ and positive otherwise.
If $m > 3$ or $1 < m < 2$ then

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^{3-m}dx = -|C|\int_{\mathbb{R}^N}|\nabla\mathcal{H}(u)|^2dx.$$

If $2 < m < 3$ then

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^{3-m}dx = |C|\int_{\mathbb{R}^N}|\nabla\mathcal{H}(u)|^2dx,$$

or equivalently

$$\frac{d}{dt}\int_{\mathbb{R}^N} u - u^{3-m}dx = -|C|\int_{\mathbb{R}^N}|\nabla\mathcal{H}(u)|^2dx.$$

• **Second energy estimate:**

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|\mathcal{H}u(x,t)|^2dx &= \int_{\mathbb{R}^N}\mathcal{H}(u)(\mathcal{H}(u))_t dx = \int_{\mathbb{R}^N}\mathcal{K}(u)u_t dx\\ &= \int_{\mathbb{R}^N}\mathcal{K}(u)\nabla\cdot(u^{m-1}\nabla\mathcal{K}(u))dx = -\int_{\mathbb{R}^N}u^{m-1}|\nabla\mathcal{K}(u)|^2dx.\end{aligned} \tag{4.7}$$

• **L^∞ estimate:** We prove that the $L^\infty(\mathbb{R}^N)$ norm does not increase in time. Indeed, at a point of maximum x_0 of u at time $t = t_0$, we have

$$u_t = (m-1)u^{m-1}\nabla u\cdot\nabla p + u^{m-1}\Delta\mathcal{K}(u).$$

The first term is zero since $\nabla u(x_0,t_0) = 0$. For the second one we have $-\Delta\mathcal{K} = (-\Delta)(-\Delta)^{-s} = (-\Delta)^{1-s}$ so that

$$\Delta\mathcal{K}u(x_0,t_0) = -(-\Delta)^{1-s}u(x_0,t_0) = -c\int_{\mathbb{R}^N}\frac{u(x_0,t_0)-u(y,t_0)}{|x_0-y|^{N-2(1-s)}}dy \le 0,$$

where $c = c(s,n) > 0$. We conclude by the positivity of u that

$$u_t(x_0,t_0) = u^{m-1}(x_0,t_0)\Delta\mathcal{K}u(x_0,t_0) \le 0.$$

• **Conservation of positivity:** we prove that if $u_0 \ge 0$ then $u(t) \ge 0$ for all times. The argument is similar to the one above.

• L^p **estimates for** $1 < p < \infty$**.** The following computations are valid for all $m \ge 1$, since $p+m-2>0$:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} u^P(x,t)dx = p\int_{\mathbb{R}^N} u^{p-1}\nabla\cdot(u^{m-1}\nabla\mathcal{K}(u))dx \\
= &\ -p\int_{\mathbb{R}^N} u^{m-1}\nabla(u^{p-1})\cdot\nabla\mathcal{K}(u)dx = -\frac{p(p-1)}{m+p-2}\int_{\mathbb{R}^N}\nabla(u^{p+m-2})\cdot\nabla\mathcal{K}(u)dx \\
= &\ \frac{p(p-1)}{m+p-2}\int_{\mathbb{R}^N} u^{p+m-2}\Delta\mathcal{K}(u)dx = -\frac{p(p-1)}{m+p-2}\int_{\mathbb{R}^N} u^{p+m-2}(-\Delta)^{1-s}u\,dx \\
\le &\ -\frac{4p(p-1)}{(m+p-1)^2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{1-s}{2}}u^{\frac{m+p-1}{2}})\right|^2 dx,
\end{aligned}$$

where we applied the Stroock-Varopoulos inequality (2.32) with $r = m+p+1$. We obtain that $\int_{\mathbb{R}^N} u^p(t)dx$ is non-increasing in time. Moreover, by Sobolev Inequality (2.34) applied to the function $f = u^{\frac{m+p-1}{2}}$, we obtain that

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^P(x,t)dx \le -\frac{4p(p-1)}{(m+p-1)^2\mathcal{S}_{1-s}^2}\left(\int_{\mathbb{R}^N}|u(x,t)|^{\frac{N(m+p-1)}{N-2+2s}}dx\right)^{\frac{N-2+2s}{N}},$$

with the restriction of $s > 1/2$ if $N = 1$.

4.3 Existence. Smooth approximate solutions

Our aim is to solve the initial-value problem (4.1)-(4.2) posed in $Q = \mathbb{R}^N \times (0,\infty)$ or at least $Q_T = \mathbb{R}^N \times (0,T)$, with parameter $0 < s < 1$. We will consider initial data $u_0 \in L^1(\mathbb{R}^N)$. We assume for technical reasons that u_0 is bounded and we also impose decay conditions as $|x| \to \infty$.

We make an approach to problem (4.1) based on regularization, elimination of the degeneracy and reduction of the spatial domain. Once we have solved the approximate problems, we derive estimates that allow us to pass to the limit in all the steps one by one, to finally obtain a weak solution of the original problem. In doing this we follow the outline of [39]. Specifically, for small $\epsilon,\delta,\mu \in (0,1)$ and $R > 0$ we consider the following initial boundary value problem posed in $Q_{T,R} = \{x \in B_R(0),\ t \in (0,T)\}$

$$\begin{cases} u_t = \delta\Delta u + \nabla(d_\mu(u)\nabla\mathcal{K}_\epsilon(u)) & \text{for } (x,t) \in Q_{T,R}, \\ u(x,0) = \widehat{u}_0(x) & \text{for } x \in B_R(0), \\ u(x,t) = 0 & \text{for } x \in \partial B_R(0),\ t \ge 0. \end{cases} \tag{4.8}$$

We explain in detail the regularization tools that we use.

• $\widehat{u}_0 = \widehat{u}_{0,\epsilon,R}$ is a nonnegative, smooth and bounded approximation of the initial data u_0 such that $\widehat{u}_0 \le u_0$ for all $\epsilon > 0$.

• For every $\mu > 0$, $d_\mu : [0,\infty) \to [0,\infty)$ is a continuous function defined by

$$d_\mu(v) = (v+\mu)^{m-1}. \tag{4.9}$$

• We consider the approximation $\mathcal{K}_\epsilon$ as follows. Let $K(z) = C_{N,s}|z|^{-(N-2s)}$ the kernel of the Riesz potential $(-\Delta)^{-s}$, $0 < s < 1$. Let $\rho_\epsilon = \epsilon^{-N}\rho(x/\epsilon)$, $\epsilon > 0$ a standard mollifying sequence, where ρ is positive, radially symmetric and decreasing, $\rho \in C_c^\infty$ and $\int_{\mathbb{R}^N} \rho dx = 1$. We define the regularization of K as $K_\epsilon = \rho_\epsilon \star K$. Then

$$\mathcal{K}_\epsilon u = K_\epsilon \star u$$

is an approximation of the Riesz potential $\mathcal{K} = (-\Delta)^{-s}$. Moreover, $\mathcal{K}$ and $\mathcal{K}_\epsilon$ are self-adjoint operators with $\mathcal{K} = \mathcal{H}^2$, $\mathcal{K}_\epsilon = \mathcal{H}_\epsilon^2$. Also, $\rho = \sigma * \sigma$ where σ has the same properties as ρ. Then, we can write $\mathcal{H}_\epsilon$ as the operator with kernel $K_{s/2} * \sigma$. That is:

$$\int_{\mathbb{R}^N} u\ \mathcal{K}_\epsilon(u)dx = \int_{\mathbb{R}^N} |\mathcal{H}_\epsilon u|^2 dx.$$

Also $\mathcal{H}_\epsilon$ commutes with the gradient:

$$\nabla \mathcal{H}_\epsilon u = \mathcal{H}_\epsilon(\nabla u).$$

The existence and uniqueness of a solution $u(x,t) = u_{\epsilon,\delta,\mu,R}(x,t)$ to Problem (4.8) is more or less standard and the solution is smooth. In the weak formulation we have

$$\int_0^T \int_{B_R} u(\phi_t - \delta\Delta\phi)dxdt - \int_0^T \int_{B_R} d_\mu(u)\nabla\mathcal{K}_\epsilon(u)\nabla\phi dxdt + \int_{B_R} \widehat{u}_0(x)\phi(x,0)dx = 0 \tag{4.10}$$

valid for smooth test functions ϕ that vanish on the spatial boundary ∂B_R and for large t. We use the notation $B_R = B_R(0)$.

4.3.1 A-priori estimates for the approximate problem

We derive suitable a-priori estimates for the solution $u_{\epsilon,\delta,\mu,R}(x,t)$ to Problem (4.8).

• **Decay of total mass:** Since $u \ge 0$ and $u = 0$ in ∂B_R, then $\dfrac{\partial u}{\partial n} \le 0$ and so, an easy computation gives us

$$\begin{aligned} \frac{d}{dt}\int_{B_R} u(x,t)dx &= \delta \int_{B_R} \Delta u\ dx + \int_{B_R} \nabla(d_\mu(u)\nabla\mathcal{K}_\epsilon(u))dx \\ &= \int_{\partial B_R} \frac{\partial u}{\partial n} d\sigma + \int_{\partial B_R} d_\mu(u)\frac{\partial(\mathcal{K}_\epsilon(u))}{\partial n} d\sigma \le 0. \end{aligned} \tag{4.11}$$

We conclude that

$$\int_{B_R} u(x,t)dx \le \int_{B_R} \widehat{u}_0(x) \quad \text{for all } t > 0.$$

• **Conservation of non-negativity**: $u(x,t) \ge 0$ for all $t > 0$, $x \in B_R$.

• **Conservation of L^∞ bound**: we prove that $0 \le u(x,t) \le ||\widehat{u}_0||_\infty$. The argument is as in the previous section, using also that at a minimum point $\Delta u \ge 0$ and at a maximum point $\Delta u \le 0$. Also at this kind of points we have that

$$\nabla d_\mu(u) = d'_\mu(u)\nabla u = 0.$$

• **Estimating the $L^p(B_R)$ norm.** We have:

$$\begin{aligned}
\frac{d}{dt}\int_{B_R} u^p dx &= p\int_{B_R} u^{p-1}(\delta\Delta u + \nabla.(d_\mu(u)\nabla\mathcal{K}_\epsilon(u)))dx\\
&= -p(p-1)\delta\int_{B_R} u^{p-2}|\nabla u|^2 dx + p\delta\int_{\partial B_R} u^{p-1}\frac{\partial u}{\partial n}ds(x)-\\
&- p(p-1)\int_{B_R} u^{p-2} d_\mu(u)\nabla u\cdot\nabla\mathcal{K}_\epsilon(u)dx + p\int_{\partial B_R} d_\mu(u)u^{p-1}\frac{\partial\mathcal{K}_\epsilon(u)}{\partial n}ds(x).
\end{aligned}$$

The boundary integrals are zero since $u = 0$ on ∂B_R. For the third integral we make the following formal estimate:

$$\begin{aligned}
III &= \int_{B_R} u^{p-2} d_\mu(u)\nabla u\cdot\nabla\mathcal{K}_\epsilon(u)dx\\
&= \int_{B_R} \nabla B(u)\cdot\nabla\mathcal{K}_\epsilon(u)dx\\
&= \int_{B_R} B(u)(-\Delta)\mathcal{K}_\epsilon(u)dx + \int_{\partial B_R} B(u)\cdot\frac{\partial\mathcal{K}_\epsilon(u)}{\partial n}ds(x)\\
&\sim \int_{B_R} B(u)(-\Delta)^{1-s}(u)dx + \int_{\partial B_R} B(u)\cdot\frac{\partial\mathcal{K}_\epsilon(u)}{\partial n}ds(x)\\
&\ge \int_{B_R} |(-\Delta)^{\frac{1-s}{2}}(\psi(u))|^2 dx.
\end{aligned}$$

First, we used the fact that $\mathcal{K}_\epsilon$ approximates the inverse fractional Laplacian $(-\Delta)^{-s}$. Then, we used the generalized Stroock-Varoupoulos Inequality (2.33) in the following context: the functions ψ and Ψ are such that $\psi' = (\Psi')^2$ and $\nabla\psi(u) = u^{p-2}(u+\mu)^{m-1}\nabla u$. We give now the explicit form of these functions:

$$\psi(z) = \int_0^z \zeta^{p-2}(\zeta+\mu)^{m-1}d\zeta, \quad \Psi(z) = \int_0^z \zeta^{\frac{p-2}{2}}(\zeta+\mu)^{\frac{m-1}{2}}d\zeta.$$

Remark that $\psi(u) = 0$ on the boundary ∂B_R. In the limit as $\mu \to 0$ we get

$$\psi(z) \to \frac{1}{p+m-2}z^{p+m-2}, \quad \Psi(z) \to \frac{2}{p+m-1}z^{\frac{p+m-1}{2}}.$$

Idea: The rigorous proof of $III \geq 0$ is based on the paper of Varopoulos [101]. This is still a work in progress and will appear in the forthcoming paper [95].

We conclude that the $L^p(B_R)$ norm of $u_\epsilon(\cdot,t)$ is decreasing in time, for small values of ϵ, that is,

$$\int_{B_R} |u_\epsilon(x,t)|^p dx \leq \int_{B_R} |u_0(x)|^p dx, \quad \forall t > 0.$$

As a consequence, $u_\epsilon(\cdot,t) \in L^p(B_R)$ for all times $t > 0$ and there exist bounds for this norm independent of the parameters δ, ϵ, μ and R if we start with initial data $u_0 \in L^p(\mathbb{R}^N)$:

$$\int_{B_R} |u_\epsilon(x,t)|^p dx \leq \int_{\mathbb{R}^N} |u_0(x)|^p dx, \quad \forall t > 0.$$

• **First energy estimate.** We choose a function F_μ such that

$$F_\mu(0) = F'_\mu(0) = 0 \quad \text{and} \quad F''_\mu(u) = 1/d_\mu(u).$$

Then, with these conditions one can see that $F_\mu(z) > 0$ for all $z > 0$. Also $F_\mu(u)$ and $F'_\mu(u)$ vanish on $\partial B_r \times [0,T]$, therefore, after integrating by parts, we get

$$\frac{d}{dt}\int_{B_R} F_\mu(u)dx = -\delta \int_{B_R} \frac{|\nabla u|^2}{d_\mu(u)}dx - \int_{B_R} |\nabla \mathcal{H}_\epsilon(u)|^2 dx, \tag{4.12}$$

where $\mathcal{H}_\epsilon = \mathcal{K}_\epsilon^{1/2}$. This formula implies that for all $0 < t < T$ we have

$$\int_{B_R} F_\mu(u(t))dx + \delta \int_0^t \int_{B_R} \frac{|\nabla u|^2}{d_\mu(u)}dxdt + \int_0^t \int_{B_R} |\nabla \mathcal{H}_\epsilon(u)|^2 dxdt = \int_{B_R} F_\mu(\widehat{u}_0)dx. \tag{4.13}$$

This implies estimates for $|\nabla \mathcal{H}_\epsilon(u)|^2$ and $\delta |\nabla u|^2/d_\mu(u)$ in $L^1(Q_{T,R})$. We investigate how the bounds for such norms depend on the parameters ϵ, δ, R, T and μ.

The explicit formula for F_μ is as follows:

$$F_\mu(u) = \begin{cases} \dfrac{1}{(2-m)(3-m)}[(u+\mu)^{3-m} - \mu^{3-m}] - \dfrac{1}{2-m}\mu^{2-m}u & \text{for } m \neq 2,3, \\ -\log(1 + (u/\mu)) + u/\mu, & \text{for } m = 3, \\ (u+\mu)\log(1 + (u/\mu)) - u, & \text{for } m = 2. \end{cases}$$

For $m = 2$ see [39].

From the formula (4.12) we obtain that the quantity $\int_{B_R} F_\mu(u(x,t))dx$ is non-increasing in time:

$$0 \leq \int_{B_R} F_\mu(u(x,t))dx \leq \int_{B_R} F_\mu(\widehat{u}_0)dx, \quad \forall t > 0.$$

Then, if we control the term $\int_{B_R} F_\mu(\widehat{u}_0)dx$, we will obtain uniform estimates independent of time $t > 0$ for the quantity

$$\delta \int_0^t \int_{B_R} \frac{|\nabla u|^2}{d_\mu(u)}dxdt + \int_0^t \int_{B_R} |\nabla \mathcal{H}_\epsilon(u)|^2 dxdt.$$

These estimate continue differently depending on the range of parameters m.

• **Uniform Bounds in the case** $m \in (1,2)$. We obtain uniform bounds in all parameters ϵ, R, δ, μ for the energy estimate (4.13), that allow us to pass to the limit and obtain a solution of the original problem (4.1). By the Mean Value Theorem

$$\begin{aligned}\int_{B_R} F_\mu(\widehat{u}_0)dx &\le \frac{1}{(2-m)(3-m)}\int_{B_R}[(\widehat{u}_0+\mu)^{3-m}-\mu^{3-m}]dx\\ &\le \frac{1}{2-m}\int_{B_R}(\widehat{u}_0+\mu)^{2-m}\widehat{u}_0 dx\\ &\le \frac{1}{2-m}(\|u_0\|_\infty+1)^{2-m}\int_{\mathbb{R}^N} u_0 dx.\end{aligned}$$

Our main estimate in the case $m \in (1,2)$ is:

$$\delta\int_0^t\int_{B_R}\frac{|\nabla u|^2}{d_\mu(u)}dxdt+\int_0^t\int_{B_R}|\nabla\mathcal{H}_\epsilon(u)|^2 dxdt \le C_1, \tag{4.14}$$

where $C_1 = C_1\,(m,u_0) = \dfrac{1}{(2-m)}(\|u_0\|_\infty+1)^{2-m}\|u_0\|_{L^1(\mathbb{R}^N)}$. This is a bound independent of the parameters ϵ, δ, R and μ.

• **Bounds in the case** $m \in (2,3)$.

$$\begin{aligned}\int_{B_R} F_\mu(\widehat{u}_0)dx &= -\frac{1}{(m-2)(3-m)}\int_{B_R}[(\widehat{u}_0+\mu)^{3-m}-\mu^{3-m}]dx+\frac{1}{m-2}\mu^{2-m}\int_{B_R}\widehat{u}_0 dx\\ &\le \frac{1}{m-2}\mu^{2-m}\int_{B_R}\widehat{u}_0 dx \le \frac{1}{m-2}\mu^{2-m}\int_{\mathbb{R}^N} u_0 dx.\end{aligned}$$

• **Bounds in the case** $m = 3$.

$$\int_{B_R} F_\mu(\widehat{u}_0)dx = -\int_{B_R}\log\left(1+\frac{\widehat{u}_0}{\mu}\right)+\int_{B_R}\frac{\widehat{u}_0}{\mu}dx \le \frac{1}{\mu}\int_{\mathbb{R}^N} u_0 dx.$$

• **Bounds in the case** $m \in (3,\infty)$.

$$\begin{aligned}\int_{B_R} F_\mu(\widehat{u}_0)dx &= \frac{1}{(m-2)(m-3)}\int_{B_R}[(\widehat{u}_0+\mu)^{3-m}-\mu^{3-m}]dx+\frac{1}{m-2}\mu^{2-m}\int_{B_R}\widehat{u}_0 dx\\ &\le \frac{1}{m-2}\mu^{2-m}\int_{B_R}\widehat{u}_0 dx \le \frac{1}{m-2}\mu^{2-m}\int_{\mathbb{R}^N} u_0 dx.\end{aligned}$$

Remark. The bounds for $m > 2$ are independent of ϵ, δ and R, but depend on the parameter μ. If $\mu \to 0$ then the bounds blow up.

• **Exponential tail in the case** $m \in (2,\infty)$. This case is more delicate since the term $\int_{B_R} F_\mu(u(t))dx$ can not be easily uniformly controlled in $\mu > 0$.

In [39], when $m = 2$, the authors prove an exponential tail control of the approximate solution by using a comparison method with a suitable family of barrier functions, called true supersolutions. In Section 4.4 we show that their proof can be adapted to the case $m \in (2, \infty)$ with a series of technical modifications caused by the power u^{m-1}.

Idea. For $m \in (2,3)$ we have

$$\frac{1}{(2-m)(3-m)} \int_{\mathbb{R}^N} \left[(u(t)+\mu)^{3-m} - \mu^{3-m} \right] dx \leq \frac{1}{(2-m)(3-m)} \int_{\mathbb{R}^N} u(t)^{3-m} dx.$$

We will prove that this quantity is finite due to the tail control result of Section 4.4.

4.3.2 Passing to the limit in the approximate problem

Functional setting. We work in the functional settings stated in Section 2.5.3.

4.3.2.1 Limit as $\epsilon \to 0$

We begin with the limit as $\epsilon \to 0$ in order to obtain a solution of the equation

$$u_t = \delta \Delta u + \nabla \cdot (d_\mu(u) \nabla \mathcal{K}(u)). \tag{4.15}$$

Let $u_\epsilon := u_{\epsilon,\delta,\mu,R}$ be the solution of (4.8). We fix δ, μ and R and we argue for ϵ close to 0. Then, by the energy formula (4.14) and the estimates from Section 4.3.1 we obtain that

$$\int_0^t \int_{B_R} \frac{|\nabla u_\epsilon|^2}{(u+\mu)^{m-1}} dxdt \leq C(\mu, m, u_0), \quad \int_0^t \int_{B_R} |\nabla \mathcal{H}_\epsilon(u_\epsilon)|^2 dxdt \leq C(\mu, m, u_0), \tag{4.16}$$

valid for all $\epsilon > 0$. Since $\|u_\epsilon\|_\infty \leq \|u_0\|_\infty$ for all $\epsilon > 0$, then

$$\int_0^t \int_{B_R} |\nabla u_\epsilon|^2 dxdt \leq C(\mu, m, u_0)(\|u_0\|_\infty + 1)^{m-1}, \quad \forall \epsilon > 0.$$

We recall that in the case $m \in (1,2)$ the constant $C(\mu, m, u_0)$ is independent of μ.

I. Convergence as $\epsilon \to 0$. We perform an analysis of the family of approximate solutions (u_ϵ) in order to derive a compactness property in suitable function spaces.

- Uniform boundedness: $u_\epsilon \in L^\infty_{x,t}(Q_{T,R})$, as we have proved.
- Gradient estimates. From the energy formula (4.16) we derive

$$u_\epsilon \in L^2_t([0,T] : H^1_{0,x}(B_R)), \quad \nabla \mathcal{H}_\epsilon(u_\epsilon) \in L^2_t([0,T] : L^2_x(\mathbb{R}^N))$$

uniformly bounded for $\epsilon > 0$. Since $\nabla \mathcal{H}_\epsilon(u_\epsilon)$ is "a derivative of order $1-s$ of u_ϵ", we conclude that

$$u_\epsilon \in L^2_t([0,T], H^{1-s}_x(\mathbb{R}^N)). \tag{4.17}$$

By potential theory arguments it follows that $\mathcal{K}_\epsilon(u_\epsilon) \in L^2_t([0,T] : H^{1+s}_x(\mathbb{R}^N))$.

• Estimates on the time derivative $(u_\epsilon)_t$: we use the equation (4.8) to obtain that

$$(u_\epsilon)_t \in L^2_t([0,T] : H_x^{-1+s}(\mathbb{R}^N)). \tag{4.18}$$

Indeed, we have:

(a) Derivatives of order $-1+s$ of Δu_ϵ are derivatives of order $1+s$ of u_ϵ, which we want to control in the L^2 norm. From the previous estimates we know that $u_\epsilon \in H_x^{1-s}(\mathbb{R}^N)$ and by the theory of fractional Sobolev spaces (see [54]) we have that $H^{1-s}(\mathbb{R}^N) \subset H^{1+s}(\mathbb{R}^N)$, which allows us to conclude that $\Delta u_\epsilon \in H_x^{-1+s}(\mathbb{R}^N)$.

(b) Derivatives of order $-1+s$ of $\Delta \mathcal{K}_\epsilon(u_\epsilon)$ are derivatives of order $1+s$ of $\mathcal{K}_\epsilon(u_\epsilon)$, which are controlled in L^2 norm, since $\mathcal{K}_\epsilon(u_\epsilon) \in L^2_t([0,T] : H_x^{1+s}(\mathbb{R}^N))$.

Now, the convergences (4.17) and (4.18), allow us to apply the compactness criteria of Simon, see Lemma 4.7.4 in the Appendix, in the context of

$$H^{1-s}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset H^{-1+s}(\mathbb{R}^N),$$

and we conclude that the family of approximate solutions (u_ϵ) is relatively compact in $C([0,T] : L^2(\mathbb{R}^N))$. Therefore there exists a limit $u_\epsilon(x,t) \to u(x,t)$ in $C([0,T] : L^2(\mathbb{R}^N))$, up to subsequences. Note that, since u_ϵ is a family of positive functions defined on B_R and extended to 0 in $\mathbb{R}^N \setminus B_R$, then the limit $u = 0$ a.e. on $\mathbb{R}^N \setminus B_R$. We will take $u = 0$ on $\mathbb{R}^N \setminus B_R$. We obtain that

$$u_\epsilon(x,t) \to u(x,t) \quad \text{in } C([0,T] : L^2(B_R)). \tag{4.19}$$

From the estimate (4.18), we obtain the weak convergence (up to subsequences) of $\nabla \mathcal{H}_\epsilon(u_\epsilon) \to w$ in $L^2_x(B_R)$. One can prove that $w = \nabla \mathcal{H} u$. A similar result holds for the convergence of $\nabla \mathcal{K}_\epsilon(u_\epsilon)$. Therefore when $\epsilon \to 0$

$$\nabla \mathcal{H}_\epsilon(u_\epsilon) \to \nabla \mathcal{H} u, \quad \nabla \mathcal{K}_\epsilon(u_\epsilon) \to \nabla \mathcal{K} u \quad \text{weakly in } L^2_x(B_R). \tag{4.20}$$

II. The limit is a solution of the new problem. We prove that the limit $u(x,t)$ of the solutions $u_\epsilon(x,t)$ is a solution of Problem (4.15). More exactly, we pass to the limit as $\epsilon \to 0$ in the definition (4.10) of a weak solution of Problem (4.8).

The convergence of the first integral in (4.10) is justified by (4.19). Now, the convergence of the second integral follows by (4.20) and then

$$\int_0^T \int_{B_R} d_\mu(u_\epsilon) \nabla \mathcal{K}_\epsilon(u_\epsilon) \nabla \phi \, dx dt \to \int_0^T \int_{B_R} d_\mu(u) \nabla \mathcal{K}(u) \nabla \phi \, dx dt, \quad \text{as } \epsilon \to 0.$$

The conclusion is that we have obtained a weak solution of the initial value problem (4.15) posed in $B_R \times [0,\infty]$ with homogeneous Dirichlet boundary conditions. The regularity of u, $\mathcal{H}(u)$ and $\mathcal{K}(u)$ is as stated before. We also have the energy formula

$$\int_{B_R} F_\mu(u(t))dx + \delta \int_0^t \int_{B_R} \frac{|\nabla u|^2}{d_\mu(u)} dxdt + \int_0^t \int_{B_R} |\nabla \mathcal{H}(u)|^2 dxdt = \int_{B_R} F_\mu(u_0)dx. \quad (4.21)$$

We do not pass now to the limit as $\delta \to 0$, because we lose H^1 estimates for u and we deal with the problem caused by the boundary data. Therefore we keep the term $\delta\Delta u$.

4.3.2.2 Limit as $R \to \infty$

We will now pass to the limit as $R \to \infty$. Since the estimates from Section 4.3.1 are uniform for all $R > 0$, then we can easy make $R \to \infty$ and obtain a solution of the problem in the whole space,

$$u_t = \delta\Delta u + \nabla \cdot \left((u+\mu)^{m-1}\nabla\mathcal{K}(u)\right) \quad x \in \mathbb{R}^N, \quad t > 0. \quad (4.22)$$

This problem satisfies the property of conservation of mass, that we prove next.

Lemma 4.3.1. *Under the assumption that $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the constructed non-negative solution of Problem* (4.22) *satisfies*

$$\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx \text{ for all } t > 0. \quad (4.23)$$

Proof. Assume $s < 1/2$ if $N = 1$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ a cutoff test function supported in the ball B_{2R} and such that $\varphi \equiv 1$ for $|x| \le R$. We get

$$\int_{B_{2R}} u_t\varphi dx = \delta \int_{B_{2R}} u\Delta\varphi dx - \int_{B_{2R}} (u+\mu)^{m-1}(\nabla\mathcal{K}(u)\cdot\nabla\varphi)dx = I_1 + I_2.$$

Since $u(t) \in L^1(\mathbb{R}^N)$ we estimate the first integral as $I_1 = O(R^{-2})$ and then $I_1 \to 0$ as $R \to \infty$. For the second integral we have

$$I_2 = \int_{B_{2R}} \mathcal{K}(u)\nabla\cdot\left((u+\mu)^{m-1}\nabla\varphi\right)dx.$$

$$I_2 = (m-1)\int_{B_{2R}} \mathcal{K}(u)(u+\mu)^{m-2}\nabla u\cdot\nabla\varphi dx + \int_{B_{2R}} \mathcal{K}(u)(u+\mu)^{m-1}\Delta\varphi dx = I_{21} + I_{22}.$$

Since $\nabla u \in L^2$ and $u \in L^\infty$,

$$|I_{21}| \le C||(u+\mu)^{m-2}||_\infty \left(\int_{B_{2R}} |\nabla u|^2 dx\right)^{1/2} \left(\int_{B_{2R}} |\mathcal{K}(u)|^2|\nabla\varphi|^2 dx\right)^{1/2}.$$

Now $\nabla\varphi = O(R^{-1})$, $\nabla\varphi \in L^p$ with $p > N$, so we need $\mathcal{K}(u) \in L^q$ for $q < 2\dfrac{1}{1-\frac{1}{N/2}} = \dfrac{2N}{N-2}$ which is true since $\mathcal{K}(u) \in L^q$ for $q > q_0 = N/(N-2s)$, and $q_0 < 2N/(N-2)$ if

$4s < N+2$. So, since $p > N$,

$$\begin{aligned} |I_{21}| &\leq C\left(\int_{B_{2R}} |\nabla \mathcal{K}(u)|^q dx\right)^{1/q} \left(\int_{B_{2R}} |\nabla \varphi|^p dx\right)^{1/p} \\ &\leq C\left(\int_{B_{2R}} R^p dx\right)^{1/p} \leq C\left(R^{N-p}\right)^{1/p} = CR^{\frac{N}{p}-1} \overset{R\to\infty}{\longrightarrow} 0. \end{aligned}$$

For I_{22}, we will use the same trick of the previous section,

$$I_{22} = \int_{B_{2R}} \mathcal{K}(u)\left[(u+\mu)^{m-1} - \mu^{m-1}\right]\Delta\varphi dx + \mu^{m-1}\int_{B_{2R}} \mathcal{K}(u)\Delta\varphi dx = I_{221} + I_{222}.$$

Now,

$$I_{222} = \mu^{m-1}\int_{B_{2R}} u\mathcal{K}(\Delta\varphi)dx = \mu^{m-1}||u||_1 O(R^{-2(1-s)}) \overset{R\to\infty}{\longrightarrow} 0,$$

where we use the fact that $\mathcal{K}\Delta$ has homogeneity $2(1-s)$ as a differential operator. Also,

$$I_{221} = \int_{B_{2R}} f'(\xi)u\mathcal{K}(u)\Delta\varphi dx,$$

where $f(x) = x^{m-1}$ and $\xi \in [\mu, \mu + u(x)]$. Again, since $u \in L^\infty$, there exists a global bound for $f(\xi)$ and so integral $I_{221} \to 0$ as $R \to \infty$ (details could be found in [39]).

In the limit $R \to \infty$, $\varphi \equiv 1$ and we get (4.23). □

Consequence. The estimates done in Section 4.3.1 can be improved when passing to the limit $R \to \infty$, since the conservation of mass (4.23) eliminates some of the integrals that presented difficulties when trying to obtain upper bounds independent of μ. Therefore, we compute the following terms in the energy estimate (4.13).

For $m \neq 2, 3$ we have

$$\begin{aligned} &\int_{B_R} F_\mu(u_0)dx - \int_{B_R} F_\mu(u)dx = \\ &= C\int_{B_R} [(u_0+\mu)^{3-m} - \mu^{3-m}]dx - \frac{1}{2-m}\mu^{2-m}\int_{B_R} u_0 dx \\ &- C\int_{B_R} [(u+\mu)^{3-m} - \mu^{3-m}]dx + \frac{1}{2-m}\int_{B_R} u dx \\ &\longrightarrow \quad C\int_{\mathbb{R}^N} [(u_0+\mu)^{3-m} - \mu^{3-m}]dx - C\int_{\mathbb{R}^N} [(u_0+\mu)^{3-m} - \mu^{3-m}]dx, \end{aligned} \tag{4.24}$$

as $R \to \infty$. We use the notation $C = \frac{1}{(2-m)(3-m)}$.

For $m = 3$ we have

$$
\begin{aligned}
&\int_{B_R} F_\mu(u_0)dx - \int_{B_R} F_\mu(u)dx = \\
&= -\int_{B_R} \log\left(1+\frac{u_0}{\mu}\right)dx + \frac{1}{\mu}\int_{B_R} u_0 dx + \int_{B_R} \log\left(1+\frac{u}{\mu}\right)dx - \frac{1}{\mu}\int_{B_R} u dx \\
&\longrightarrow \quad \int_{\mathbb{R}^N} \log\left(1+\frac{u}{\mu}\right)dx - \int_{\mathbb{R}^N} \log\left(1+\frac{u_0}{\mu}\right)dx \quad \text{as } R\to\infty. \qquad (4.25)
\end{aligned}
$$

We summarize the results obtained until now. The following theorem is stated for the solution $u = u_{\epsilon,\delta,\mu}$ of the approximate problem in the whole space. Passing to limit $\epsilon \to 0$ and $\delta \to 0$ is as before. We do not pass to the limit now, since the solution of the approximate problem has good regularity in x and t.

Theorem 4.3.1. *Let $m > 1$ and $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be non-negative. Then there exists a weak solution $u = u_{\epsilon,\delta,\mu}$ of Problem (4.22) posed in $\mathbb{R}^N \times (0,T)$ with initial data u_0. Moreover, $u \in L^\infty(0,\infty : L^1(\mathbb{R}^N))$, and for all $t > 0$ we have*

$$\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx$$

and $||u(\cdot,t)||_\infty \le ||u_0||_\infty$. The first energy estimate holds in the form

$$
\begin{aligned}
\delta\int_0^t\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{(u+\mu)^2}dxdt \quad &+ \quad \int_0^t\int_{\mathbb{R}^N} |\nabla \mathcal{H}(u)|^2 dxdt + \int_{\mathbb{R}^N} \log\left(\frac{u_0}{\mu}+1\right)dx \\
&= \quad \int_{\mathbb{R}^N} \log\left(\frac{u(t)}{\mu}+1\right)dx \qquad (4.26)
\end{aligned}
$$

if $m = 3$ and

$$
\begin{aligned}
\delta\int_0^t\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{(u+\mu)^{m-1}}dxdt \quad &+ \quad \int_0^t\int_{\mathbb{R}^N} |\nabla \mathcal{H}(u)|^2 dxdt + \\
+C\int_{\mathbb{R}^N} \left[(u(t)+\mu)^{3-m} - \mu^{3-m}\right]dx \quad &= \quad C\int_{\mathbb{R}^N} \left[(u_0+\mu)^{3-m} - \mu^{3-m}\right]dx \quad (4.27)
\end{aligned}
$$

if $\;m \ne 2,3$, where $C = C(m) = \frac{1}{(2-m)(3-m)}$.

4.3.2.3 Limit as $\mu \to 0$

We would like to take the limit $\mu \to 0$ in terms 3 and 4 of the energy estimates (4.26) and (4.27). It is enough to find uniform bounds in $\mu > 0$ for these integrals. First we have to ensure that both terms, the one in u and the one in u_0 are finite.

• Case $m \in (1,2)$. By the mean value theorem,

$$
\begin{aligned}
\frac{1}{(m-2)(3-m)}\int_{\mathbb{R}^N} \left[(u_0+\mu)^{3-m} - \mu^{3-m}\right]dx &\le \frac{1}{(m-2)}\int_{\mathbb{R}^N} (u_0+\mu)^{2-m} u_0 dx \\
&\le \frac{(||u_0||_\infty + 1)^{2-m}}{m-2}\int_{\mathbb{R}^N} u_0 dx.
\end{aligned}
$$

This bound is independent of μ. Thus, we conclude the proof of Theorem 4.1.1 stated in the introduction of this chapter.

• Case $m \in (2,3)$. The function $f(x) = x^{3-m}$ is concave and so $f(u+\mu) \le f(\mu) + f(u)$. In this way,

$$\frac{1}{(2-m)(3-m)} \int_{\mathbb{R}^N} \left[(u(t)+\mu)^{3-m} - \mu^{3-m} \right] dx \le \frac{1}{(2-m)(3-m)} \int_{\mathbb{R}^N} u(t)^{3-m} dx.$$

We don't know a priory if the last integral is finite. We need some kind of tail control of the solution. In the following section we prove that for $m > 2$ and an initial data with exponential tail decay, we get also a solution with exponential tail decay. In this way, the last estimate is uniform in μ.

Remarks. • In the case $m = 2$ the corresponding term is $\int_{\mathbb{R}^N} u \log^-(u+\mu) dx$ which is uniformly bounded if u has an exponential tail. This has been proved by Caffarelli and Vázquez in [39]. We do not repeat the proof here.

• The case $m \ge 3$ is more difficult since we can not find uniform estimates in $\mu > 0$ for the energy estimates that allow us to pass to the limit.

4.4 Exponential tail control in the case $m \ge 2$. Existence of weak solutions for $m \in [2,3)$

In this section we will give the proof of Theorem 4.1.2. Weak solutions of the original problem are constructed by passing to the limit after a tail control step.

We develop a comparison method with a suitable family of barrier functions, that in [39] received the name of *true supersolutions*.

Theorem 4.4.1. *Let $0 < s < 1/2$, $m \ge 2$ and let u be the solution of Problem (4.22). We assume that u is bounded $0 \le u(x,t) \le L$ and that u_0 lies below a function of the form*

$$U_0(x) = Ae^{-a|x|}, \quad A, a > 0.$$

If A is large, then there is a constant $C > 0$ that depends only on (N, s, a, L, A) such that for any $T > 0$ we will have the comparison

$$u(x,t) \le Ae^{Ct-a|x|} \quad \text{for all} \quad x \in \mathbb{R}^N, \ 0 < t \le T.$$

Proof. In order to have enough regularity in the comparison argument below, we extend the proof for the solutions constructed in Theorem 4.3.1 to the whole space with parameters δ, ϵ and $\mu > 0$, and we will show that the constants in the upper estimate are uniform with respect to such parameters if the parameters are small.

• **Reduction.** By scaling we may put $a = L = 1$. This is done by considering instead of u, the function $\tilde{u}$ defined as

$$u(x,t) = L\tilde{u}(ax, bt), \quad b = L^{m-1}a^{2-2s}, \tag{4.28}$$

which satisfies the equation

$$\tilde{u}_t = \delta_1 \Delta \tilde{u} + \nabla.(\tilde{d}(\tilde{u})\nabla \mathcal{K}(\tilde{u})),$$

with $\delta_1 = a^{2s}\delta/L^{m-1}$, $\tilde{d}(\tilde{u}) = d_{\mu_1}(\tilde{u}) = (\tilde{u}+\mu_1)^{m-1}$, $\mu_1 = \mu/L$.

Note that then $\tilde{u}(x,0) \le A_1 e^{-|x|}$ with $A_1 = A/L$. The corresponding bound for $\tilde{u}(x,t)$ will be $\tilde{u}(x,t) \le A/L\ e^{C_1 t - |x|}$ with $C_1 = C/b = C\left(L^{m-1}a^{2-2s}\right)^{-1}$, or equivalently $C(a,L,A) = L^{m-1}a^{2-2s}C(1,1,A/L)$.

• **Contact analysis.** Therefore we assume that $0 \le u(x,0) \le 1$ and also that

$$u(x,0) \le Ae^{-r}, \quad r = |x| > 0,$$

where $A > 0$ is a constant that will be chosen below, say larger than 2. Given constants C, ϵ and $\eta > 0$, we consider a radially symmetric candidate for the upper barrier function of the form

$$\widehat{U}(x,t) = Ae^{Ct-r} + \epsilon A e^{\eta t},$$

and we take ϵ small. Then C will be determined in terms of A to satisfy a true supersolution condition which is obtained by contradiction at the first point (x_c, t_c) of possible contact of u and $\widehat{U}$.

The equation satisfied by u can be written in the form

$$u_t = \delta \Delta u + (m-1)(u+\mu)^{m-2}\nabla u \cdot \nabla p + (u+\mu)^{m-1}\Delta P. \tag{4.29}$$

We will obtain necessary conditions in order for equation (4.29) to hold at the contact point (x_c, t_c). Then, we prove there is a suitable choice of parameters $C, A, \eta, \epsilon, \mu$ such that the contact can not hold.

Estimates on u and p at the first contact point. For $0 < s < 1/2$, at the first contact point (x_c, t_c) we have the estimates

$$\partial_r u = -Ae^{Ct_c - r_c}, \quad \Delta u \le Ae^{Ct_c - r_c}, \quad u_t \ge ACe^{Ct_c - r_c} + \epsilon\eta A e^{\eta t_c}.$$

Since we assumed our solution u is bounded by $0 \le u \le 1$, then

$$u(x_c, t_c) = Ae^{Ct_c - r_c} + \epsilon A e^{\eta t_c} \le 1. \tag{4.30}$$

Moreover, from [39] we have the following bounds for the pressure term at the contact point:

$$\Delta P(x_c, t_c) \le K_1, \quad (-\partial_r p)(x_c, t_c) \le K_2. \tag{4.31}$$

Necessary conditions at the first contact point. Equation (4.29) at the contact point (x_c, t_c) with $r_c = |x_c|$, implies that

$$ACe^{Ct_c - r_c} + \epsilon\eta A e^{\eta t_c} \leq \delta A e^{Ct_c - r_c} + (m-1)\left(u(x_c,t_c)+\mu\right)^{m-2}(-Ae^{Ct_c-r_c})(\partial_r P) + \\ + (u(x_c,t_c)+\mu)^{m-1}\Delta P.$$

We denote $\xi := r_c + (\eta - C)t_c$. Using also (4.31) with K the maximum of the two constants, we obtain, after we simplify the previous inequality by $Ae^{Ct_c-r_c}$,

$$C + \epsilon\eta e^{\xi} \leq \delta + (m-1)\left(u(x_c,t_c)+\mu\right)^{m-2}K + (u(x_c,t_c)+\mu)^{m-2}(1+\epsilon e^{\xi} + \frac{\mu}{A}e^{r_c - Ct_c})K,$$

and equivalently

$$C + \epsilon\eta e^{\xi} \leq \delta + K\left(u(x_c,t_c)+\mu\right)^{m-2}\left(m + \epsilon e^{\xi} + \frac{\mu}{A}e^{r_c-Ct_c}\right).$$

We take $C = \eta$ and $\dfrac{\mu}{A} \leq \epsilon$. Then

$$C + \epsilon C e^{r_c} \leq \delta + K\left(u(x_c,t_c)+\mu\right)^{m-2}\left(m + \epsilon e^{r_c} + \epsilon e^{r_c - Ct_c}\right).$$

Moreover,

$$C + \epsilon C e^{r_c} \leq \delta + K\left(u(x_c,t_c)+\mu\right)^{m-2}\left(m + 2\epsilon e^{r_c}\right).$$

By (4.30) we have that

$$\mu < u(x_c,t_c) + \mu < 1 + \mu.$$

Case $m > 2$. Then

$$C + \epsilon C e^{r_c} \leq \delta + K\left(1+\mu\right)^{m-2}\left(m + 2\epsilon e^{r_c}\right).$$

This is impossible for C large enough such that

$$C \geq \delta + mK(1+\mu)^{m-2} \quad \text{and} \quad C \geq 2K\left(1+\mu\right)^{m-2}. \tag{4.32}$$

Since $\mu < 1$ and $\delta < 1$, then we can choose C as constant, only depending on m and K.

The case $m = 2$ also works.

• **Regularity of solutions.** To be rigorous, we consider a problem with smooth velocity field ∇P by regularizing the kernel:

$$P(x,t) = \int K_\epsilon(y)u(x+y)dy,$$

where $K_\epsilon(y) = K(y)$ for $\epsilon \leq |y| \leq 1/\epsilon$, $K_\epsilon(y)$ is a parabolic cap with C^1 fit in $|y| \leq \epsilon$, and finally $K_\epsilon(y) = 0$ for $|y| \leq 2/\epsilon$. The regularization mentioned in Section 4.3 will also do.

We apply the previous proof to the solutions of this approximated problem. The solutions u_ϵ to this problem have bounded speeds, they are smooth and bounded with smooth and bounded ∇P, and the previous estimates for P_r and ΔP at the contact points (x_c, t_c) hold uniformly in ϵ. Passing to the limit $\epsilon \to 0$, the previous conclusions hold for any weak limit solution as constructed above, see Equation (4.8). The extra limit $\delta \to 0$ then offers no difficulty.

□

Theorem 4.4.2. *Let $1/2 \le s < 1$, $m \ge 2$. Under the assumptions of the previous theorem, the stated tail estimate works locally in time. The global statement must be replaced by the following: there exists an increasing function $C(t)$ such that*

$$u(x,t) \le Ae^{C(t)t-a|x|} \quad \textit{for all } x \in \mathbb{R}^N \textit{ and all } 0 \le t \le T. \tag{4.33}$$

Proof. The proof of this result is similar to the one in [39] but with a technical adaptation to our model.

□

4.5 Finite propagation property for $m \ge 2$

In this section we will prove that compactly supported initial data $u_0(x)$ determine to solutions $u(x,t)$ that have the same property for all positive times.

Theorem 4.5.1. *Let $m \ge 2$. Assume u is a bounded solution, $0 \le u \le L$, of Equation (4.1) with $\mathcal{K} = (-\Delta)^{-s}$ with $0 < s < 1$ ($0 < s < 1/2$ if $N = 1$), as constructed in Theorem 4.1.2. Assume that u_0 has compact support. Then $u(\cdot, t)$ is compactly supported for all $t > 0$. More precisely, if $0 < s < 1/2$ and u_0 is below the "parabola-like" function*

$$U_0(x) = a(|x| - b)^2,$$

for some $a, b > 0$, with support in the ball $B_b(0)$, then there is a constant C large enough, such that

$$u(x,t) \le a(Ct - (|x| - b))^2.$$

Actually, we can take $C(L,a) = C(1,1)L^{m-\frac{3}{2}+s}a^{\frac{1}{2}-s}$. For $1/2 \le s < 2$ a similar conclusion is true, but $C = C(t)$ is an increasing function of t and we do not obtain a scaling dependence of L and a.

Proof. The method is similar to the tail control section. We assume $u(x,t) \ge 0$ has bounded initial data $u_0(x) = u(x,t_0) \le L$, and also that u_0 is below the parabola $U_0(x) = a(|x| - b)^2$, $a, b > 0$. Moreover the support of U_0 is the ball of radius b and the graphs of u_0 and U_0 are strictly separated in that ball. We take as comparison function $U(x,t) = a(Ct - (|x| - b))^2$ and argue at the first point in space and time where $u(x,t)$

touches U from below. The fact that such a first contact point happens for $t > 0$ and $x \neq \infty$ is justified by regularization, as before. We put $r = |x|$.

By scaling we may put $a = L = 1$. We denote by (x_c, t_c) this contact point and we have $u(x_c, t_c) = U(x_c, t_c) = (b + Ct_c - |x_c|)^2$. We examine in detail the situation in which the contact point is not the minimum of $U(x,t)$: x_c lies at a distance $h > 0$ from the front $|x_f(t_c)| := b + Ct_c$ (the boundary of the support of the parabola $U(x,t)$ at time t_c), that is

$$b + Ct_c - |x_c| = h > 0.$$

Note that since $u \leq 1$ we must have $|h| \leq 1$. Assuming that u is also C^2 smooth, since we deal with a first contact point (x_c, t_c), we have that $u = U$, $\nabla(u - U) = 0$, $\Delta(u - U) \leq 0$, $(u - U)_t \geq 0$, that is

$$u(x_c, t_c) = h^2, \quad u_r = -2h, \quad \Delta u \leq 2N, \quad u_t \geq 2Ch.$$

For $P = \mathcal{K}(u)$ and using the equation $u_t = (m-1)u^{m-2}\nabla u \cdot \nabla P + u^{m-1}\Delta P$, we get the inequality

$$2Ch \leq 2(m-1)h^{2m-3}\left(-\overline{P_r} + \frac{h}{2}\overline{\Delta P}\right), \tag{4.34}$$

where $\overline{P_r}$ and $\overline{\Delta P}$ are the values of P_r and ΔP at the point (x_c, t_c). In order to get a contradiction, we will use estimates for the values of $\overline{P_r}$ and $\overline{\Delta P}$ already proved in [39] (see Theorem 5.1. of [39])

$$-\overline{P_r} \leq K_1 + K_2 h^{1+2s} + K_3 h, \quad \overline{\Delta P} \leq K_4. \tag{4.35}$$

Therefore, inequality (4.34) combined with the estimates (4.35) implies that

$$2C \leq 2(m-1)h^{2m-4}\left(K_1 + K_2 h^{1+2s} + Kh\right), \tag{4.36}$$

which is impossible for C large (independent of h), since $m > 2$ and $|h| \leq 1$. Therefore, there cannot be a contact point with $h \neq 0$. In this way we get a minimal constant $C = C(N, s)$ for which such contact does not take place.

Remark: For $m < 2$, we do not obtain a contradiction in the estimate (4.36), since the term $K_1 h^{2m-4}$ can be very large for small values of $|h|$.

• **Reduction. Dependence on L and a.** The equation is invariant under the scaling

$$\widehat{u}(x,t) = Au(Bx, Tt) \tag{4.37}$$

with parameters $A, B, T > 0$ such that $T = A^{m-1}B^{2-2s}$.

Step I. We prove that if u has height $0 \leq u(x,t) \leq 1$ and initially satisfies $u(x,0) = u_0(x) \leq (|x| - b)^2$ then $u(x,t) \leq U(x,t) = (Ct - (|x| - b))^2$ for all $t > 0$.

STEP II. We search for parameters $A,\ B,\ T$ for which the function $\widehat{u}$ is defined by (4.37) satisfies

$$0 \le \widehat{u}(x,t) \le L, \quad \widehat{u}(x,0) \le \widehat{a}(|x| - \widehat{b})^2.$$

An easy computation gives us

$$A = L, \quad AB^2 = \widehat{a}, \quad \widehat{b} = b/B.$$

Moreover, by the relation between A, B and T we obtain $A = L$, $B = (\widehat{a}/L)^{1/2}$ and then $T = L^{m-2+s}\widehat{a}^{1-s}$. Then $\widehat{u}(x,t)$ is below the upper barrier $\widehat{U}(x,t) = \widehat{a}(\widehat{C}t - (|x| - \widehat{b}))^2$ where the new speed is given by

$$\widehat{C} = CA^{m-1}B^{1-2s} = CL^{m-\frac{3}{2}+s}\,\widehat{a}^{\frac{1}{2}-s}.$$

• **Case** $1/2 \le s < 1$**.** The proof in this case is more technical and will appear in the forthcoming paper [95].

□

Lemma 4.5.1. *Under the assumptions of Theorem 4.5.1 there is no contact between $u(x,t)$ and the parabola $U(x,t)$, in the sense that strict separation of u and U holds for all $t > 0$ if C is large enough.*

Proof. We want to eliminate the possible contact of the supports at the lower part of the parabola, that is the minimum $|x| = Ct + b$. Instead of analyzing the possible contact point, we proceed by a change in the test function that we replace by

$$U_\epsilon(x,t) = \begin{cases} (Ct - (|x| - b))^2 + \epsilon(1 + Dt) & \text{for } |x| \le b + Ct, \\ \epsilon(1 + Dt), & \text{for } |x| \ge b + Ct. \end{cases}$$

The function U_ϵ is constructed from the parabola U by a vertical translation $\epsilon(1 + Dt)$ and a lower truncation with $1 + Dt$ outside the ball $\{|x| \le b + Ct\}$. Here $0 < \epsilon < 1$ is a small constant and $D > 0$ will be suitable chosen.

We assume that the solution $u(x,t)$ starts as $u(x,0) = u_0(x)$ and touches for the first time the parabola U_ϵ at $t = t_c$ and spatial coordinate x_c. The contact point can not be a ball $\{|x| \le b + Ct\}$ since the U_ϵ is a parabola here and this case was eliminated in the previous Theorem 4.5.1. Consider now the case when the first contact point between $u(x,t)$ and $U_\epsilon(x,t)$ is when $|x_c| \ge b + Ct_c$. At the contact point we have that $u = U_\epsilon$, $\nabla(u - U_\epsilon) = 0$, $\Delta(u - U_\epsilon) \le 0$, $(u - U_\epsilon)_t \ge 0$. In this region the spatial derivatives of U_ϵ are zero, hence the equation gives us

$$D\epsilon = (\epsilon(1 + Dt_c))^{m-1}\overline{\Delta P}$$

where $\overline{\Delta P}$ is the value of $\Delta P = (-\Delta)^{1-s}u$ at the point (x_c, t_c). Since ϵ is small we get that the bound $u(x,t) \le U_1(x,t)$ is true for all $|x| \le \mathbb{R}^N$. This allows us to prove that

that $\overline{\Delta P}$ is bounded by a constant K. We obtain that $D\epsilon \leq (\epsilon(1+Dt_c))^{m-1}K$. Since $m \geq 2$ and $\epsilon < 1$, this implies that

$$D \leq (1+Dt_c)^{m-1}K.$$

We obtain a contradiction for large D, for example $D = 2K$, and for

$$t_c < T_c = \frac{1}{2K}\left(2^{1/(m-1)} - 1\right).$$

Therefore, we proved that a contact point between u and U_ϵ is not posible for $t < T_c$, and thus $u(x,t) \leq U_\epsilon(x,t)$ for $t < T_c$. The estimate on t_c is uniform in ϵ and we obtain in the limit $\epsilon \to 0$ that

$$u(x,t) \leq U(x,t) = (Ct - (|x| - b)) \quad \text{for } t < \frac{1}{2K}\left(2^{1/(m-1)} - 1\right).$$

As a consequence, the support of $u(x,t)$ is bounded by the line $|x| = Ct + b$ in the time interval $[0, T_c)$. The comparison for all times can be proved with an iteration process in time.

• Regularity requirements. Using the smooth solutions of the approximate equations, the previous conclusions hold for any weak limit solution.

□

4.5.1 Growth estimates of the support

The following result about the free boundary is valid only for $s < 1/2$ and for solutions with bounded and compactly supported initial data. The result is a direct consequence of the parabolic barrier study done in the previous section. Since that barrier does not depend explicitly on m if $m \geq 2$, the proof presented in [39] is valid here. By free boundary $\mathcal{FB}(u)$ we mean, the topological boundary of the support of the solution $S(u) = \overline{\{(x,t) : u(x,t) > 0\}}$.

Corolary 4.5.1. *Let u_0 be bounded from above by L with $u_0(x) = 0$ for $|x| > R$ for some $R > 0$. If $(x,t) \in \mathcal{FB}(u)$ then $x \leq R + Ct^{1/(2-2s)}$.*

4.5.2 Persistence of positivity

This property is also interesting in the sense that avoids the possibility of degeneracy points for the solutions. In particular, assuming that the solutions are continuous, it implies the non-shrinking of the support. Due to the nonlocal character of the operator, the following theorem can be proved only for a certain class of solutions.

Lemma 4.5.2. *Let u be a weak solution as constructed in Theorem 4.1.2 and assume that the initial data $u_0(x)$ is radially symmetric and non-increasing in $|x|$. Then $u(x,t)$ is also radially symmetric and non-increasing in $|x|$.*

The proof of this lemma is a technical result that is still a work in progress.

Theorem 4.5.2. *Let u be a weak solution as constructed in Theorem 4.1.2 and assume that it is a radial function of the space variable $u(|x|,t)$ and is non-increasing in $|x|$. If $u_0(x)$ is positive in a neighborhood of a point x_0, then $u(x_0,t)$ is positive for all times $t>0$.*

Proof. A similar technique as the one presented in the tail analysis is used for this proof, but with what we call true subsolutions. Assume $u_0(x)\ge c>0$ in a ball $B_R(x_0)$. By translation and scaling we can also assume $c=R=1$ and $x_0=0$. Again, we will study a possible first contact point with a barrier that shrinks quickly in time, like

$$U(x,t)=e^{-at}F(|x|), \tag{4.38}$$

with $F:\mathbb{R}_{\ge 0}\longrightarrow\mathbb{R}_{\ge 0}$ to be chosen later and $a>0$ large enough. Choose $F(0)=1/2$, $F(r)=0$ for $r\ge 1/2$ and $F'(r)\le 0$ for all $r\in\mathbb{R}_{\ge 0}$. The contact point (x_c,t_c) is sought in $B_{1/2}(0)\times(0,\infty)$. By approximation we can assume that u is positive everywhere so there are no contact points at the parabolic border. At the possible contact point (x_c,t_c) we have

$$u(x_c,t_c)=U(x_c,t_c),\quad u_t(x_c,t_c)\le U_t(x_c,t_c)=-aU(x_c,t_c),$$

$$\nabla u(x_c,t_c)=\nabla U(x_c,t_c)=e^{-at_c}F'(|x_c|)\mathbf{e_r},\quad \mathbf{e_r}=x_c/|x_c|.$$

We recall the equation

$$u_t=(m-1)u^{m-2}\nabla u\nabla P+u^{m-1}\Delta P.$$

Then at the contact point (x_c,t_c) we have

$$-aU=U_t\ge u_t=(m-1)U^{m-2}\nabla U\overline{\nabla P}+U^{m-1}\overline{\Delta P},$$

where $\overline{\Delta P}=\Delta P(x_c,t_c)$. Then

$$-ae^{-at_c}F(|x_c|)\ge(m-1)e^{-a(m-2)t_c}F(|x|)^{m-2}e^{-at_c}F'(|x_c|)\overline{P_r}+e^{-a(m-1)t_c}F(|x_c|)^{m-1}\overline{\Delta P}.$$

According to [39] we know that the term $F'(|x|)\,\overline{P_r}\ge 0$ and $\overline{\Delta P}$ is bounded uniformly. Therefore

$$-ae^{-at_c}F(|x_c|)\ge e^{-a(m-1)t_c}F(|x_c|)^{m-1}\overline{\Delta P}.$$

Simplifying and using that $m\ge 2$, $\overline{\Delta P}$ is bounded uniformly and also F is bounded, we obtain

$$a\le -e^{-a(m-2)t_c}F(|x_c|)^{m-2}\overline{\Delta P}\le Ke^{-a(m-2)t_c}\le K.$$

This is not true if $a>K$ and we arrive at a contradiction. $\square$

4.6 Infinite propagation speed in the case $1 < m < 2$

In this section we will consider model (4.1)

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla P), \quad P = (-\Delta)^{-s} u, \tag{4.39}$$

for $x \in \mathbb{R}$, $t > 0$ and $s \in (0, 1)$. We take compactly supported initial data $u_0 \geq 0$ such that $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Our main result is the following theorem.

Theorem 4.6.1. *Let $m \in (1, 2)$, $s \in (0, 1)$ and $N = 1$. Let u be the solution of Problem (4.39) with initial data $u_0 \geq 0$ radially symmetric and monotone decreasing in $|x|$. Then $u(x, t) > 0$ for all $t > 0$, $x \in \mathbb{R}$, that is the solution has* ***infinite speed of propagation****.*

Proof. This is a consequence of Theorem 4.6.2, where we prove that

$$v(x, t) = \int_{-\infty}^{x} u(y, t) dy > 0 \quad \text{for } t > 0,\ x \in \mathbb{R}.$$

Therefore for every $t > 0$ there exist points x arbitrary far from the origin such that $u(x, t) > 0$. Since u inherits the symmetry and monotonicity properties of the initial data by Lemma 4.5.2, this ensures that u can not take zero values.

□

Our result is new and unexpected, since is breaks the finite propagation theory developed for the case $m \geq 2$. This way we continue the theory of the porous medium equation with potential pressure, by proving that the model (4.39) has different propagation properties depending on the exponent m by the ranges $m \geq 2$ and $1 < m < 2$. Such a behaviour is well known to be typical for the classical Porous Medium Equation $u_t = \Delta u^m$, recovered for $s = 0$, which has finite propagation for $m > 1$ and infinite propagation for $m \leq 1$.

4.6.1 The integrated problem

In the one-dimensional case, equation (4.39) is related to the equation satisfied by the integrated solution v, where $v_x = u$. Therefore $v(x, t)$ is a solution of the problem

$$\partial_t v = -|v_x|^{m-1} (-\Delta)^{\alpha} v. \tag{4.40}$$

The exponents α and s are related by

$$\alpha = 1 - s. \tag{4.41}$$

Moreover, for $\alpha = 1/2$, the operator $(-\Delta)^{1/2}$ can be described as $(-\Delta)^{1/2} v(x) = \mathcal{H} v_x$, where $\mathcal{H}$ is the Hilbert transform defined by $\widehat{(\mathcal{H}u)}(\xi) = -i \ \mathrm{sgn}(\xi)\ \widehat{u}(\xi)$.

We consider Problem (4.40) with initial data

$$v(x,0)=v_0(x) \quad \text{ for all } x\in\mathbb{R}, \tag{4.42}$$

where $v_0:\mathbb{R}\to[0,\infty)$ will be such that:

$$v_0(x)=0 \text{ for } x<-\eta, \quad v_0(x)=M \text{ for } x>\eta, \quad v_0'(x)\geq 0 \text{ for } x\in(-\eta,\eta), \tag{4.43}$$

where $\eta>0$ is fixed from the beginning. This kind of assumption on the initial data v_0 is related to compactly supported data u_0 and this will be justified in the following section.

• **Connection between Model** (4.39) **and Model** (4.40)

We explain how the properties of the Model (4.39) with $N=1$ can be obtained via a study of the properties of the integrated problem (4.40). We consider Problem (4.39) with compactly supported initial data u_0 such that $u_0\geq 0$. Let us say that supp $u_0\subset[-\eta,\eta]$, where $\eta>0$. Therefore the corresponding initial data to be considered for the integrated problem is

$$v_0(x)=\int_{-\infty}^{x}u_0(y)dy, \quad \forall x\in\mathbb{R}.$$

Then, $v_0(x)$ is computed as follows (see Figure 4.2):

$$v_0(x)=\begin{cases} 0, & x<\eta,\\ \displaystyle\int_{-x_0}^{x}u_0(y)dy, & y\in[-\eta,\eta],\\ \displaystyle\int_{-\eta}^{\eta}u_0(y)dy, & y\geq\eta,. \end{cases} \tag{4.44}$$

This function clearly satisfies assumption (4.43) stated in the introduction with $M=\int_{\mathbb{R}}u_0(x)dx$ the total mass.

The property of infinite speed of propagation for model (4.39), that is $u(x,t)>0$ for all $x>0$, $t>0$, holds true whenever $v(x,t)$ is an increasing function in the space variable x. Moreover, since $u(x,t)$ enjoys the property of conservation of mass, then $v(x,t)$ satisfies (see Figure 4.2)

$$\lim_{x\to-\infty}v(x,t)=0, \quad \lim_{x\to+\infty}v(x,t)=M$$

for all $t\geq 0$. Moreover, if $t>0$ then $0<v(x,t)<M$ for all $x\in\mathbb{R}$. We devote a separate study to the solution v of the integrated problem (4.40) in Section 4.6.2.

4.6.2 Viscosity solutions

Notion of solution. We define the notions of viscosity sub-solution, super-solution and solution in the sense of Crandall-Lions [45]. The definition will be adapted to our

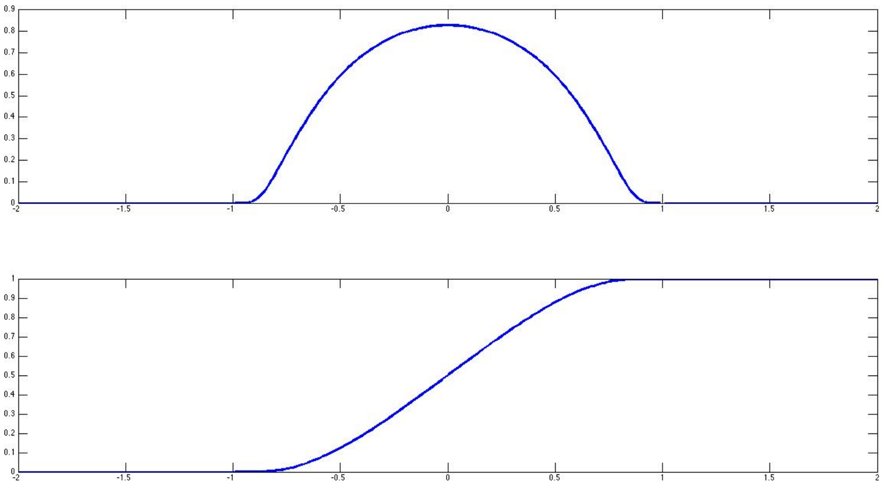


FIGURE 4.2: Typical compactly supported initial data for models (4.39) and (4.40).

problem by considering the time dependency and also the nonlocal character of the Fractional Laplacian operator. For a presentation of the theory of viscosity solutions to more general integro-differential equations we refer to Barles and Imbert [15].

It will be useful to make the notations:

$\mathrm{USC}(Q) = \{$upper semi-continuous functions $u : Q \to \mathbb{R}\}$,

$\mathrm{LSC}(Q) = \{$lower semi-continuous functions $u : Q \to \mathbb{R}\}$,

$\mathrm{C}(Q) = \{$continuous functions $u : Q \to \mathbb{R}\}$.

Definition 4.6.1. *Let $v \in USC(\mathbb{R} \times (0, \infty))$ (resp. $v \in LSC(\mathbb{R} \times (0, \infty))$). We say that v is a* ***viscosity sub-solution*** *(resp.* ***super-solution****) of equation* (4.40) *on* $\mathbb{R} \times (0, \infty)$ *if for any point (x_0, t_0) with $t_0 > 0$ and any $\tau \in (0, t_0)$ and any test function $\varphi \in C^2(\mathbb{R} \times (0, \infty)) \cap L^\infty(\mathbb{R} \times (0, \infty))$ such that $v - \varphi$ attains a global maximum (minimum) at the point (x_0, t_0) on*

$$Q_\tau = \mathbb{R} \times (t_0 - \tau, t_0]$$

we have that

$$\partial_t\varphi(x_0, t_0) + |\varphi_x(x_0, t_0)|^{m-1}((-\Delta)^\alpha \varphi(\cdot, t_0))(x_0) \le 0 \quad (\ge 0).$$

Since equation (4.40) is invariant under translation, the test function φ in the above definition can be taken such that φ touches v from above in the sub-solution case, resp. φ touches v from below in the super-solution case.

We say that v is a **viscosity sub-solution** (resp. **super-solution**) of the initial-value problem (4.40)-(4.42) on $\mathbb{R} \times (0, \infty)$ if it satisfies moreover at $t = 0$

$$v(x, 0) \le \limsup_{y \to x,\ t \to 0} v(y, t) \quad \text{(resp. } v(x, 0) \ge \liminf_{y \to x,\ t \to 0} v(y, t)).$$

We say that $v \in C(\mathbb{R} \times (0,\infty))$ is a **viscosity solution** if v is a viscosity sub-solution and a viscosity super-solution on $\mathbb{R} \times (0,\infty)$.

Existence of a unique viscosity solution of Problem (4.40)-(4.42) follows by passing to the limit in the approximate problems (called problems with vanishing viscosity) as in [22]. The limit of a sequence of viscosity solutions is a viscosity solution for our problem.

The standard comparison principle for viscosity solutions holds true. We refer to Imbert, Monneau and Rouy [71] where they treat the case $m = 2$ and $\alpha = 1/2$. Also, we mention Jakobsen and Karlsen [73] for the elliptic case.

Proposition 4.6.1. *Let $m \in (1,2)$, $\alpha \in (0,1)$, $N = 1$. Let w be a sub-solution and W be a super-solution in the viscosity sense of equation* (4.40)*. If $w(x,0) \le v_0 \le W(x,0)$, then $w \le W$ in $\mathbb{R} \times (0,\infty)$.*

We give now our extended version of parabolic comparison principle, which represents an important instrument when using barrier methods. This type of result is motivated by the nonlocal character of the problem and the construction of lower barriers in a desired region $\Omega \subset \mathbb{R}$ possibly unbounded. This determines the parabolic boundary of a domain of the form $\Omega \times [0,T]$ to be $(\mathbb{R} \setminus \Omega) \times [0,T] \cup \mathbb{R} \times \{0\}$, where $\Omega \subset \mathbb{R}$. A similar parabolic comparison has been proved in [37] and has been used for instance in [37, 97].

Proposition 4.6.2. *Let $m > 1$, $\alpha \in (0,1)$. Let v be a viscosity solution of Problem* (4.40)-(4.42)*. Let $\Phi : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ such that $\Phi \in C^2(\Omega \times (0,T))$. Assume that*

- $\Phi_t + |\Phi_x|^{m-1}(-\Delta)^\alpha \Phi < 0$ *for $x \in \Omega$, $t \in [0,T]$;*
- $\Phi(x,0) < v(x,0)$ *for all $x \in \mathbb{R}$ (comparison at initial time);*
- $\Phi(x,t) < v(x,t)$ *for all $x \in \mathbb{R} \setminus \Omega$ and $t \in (0,T)$ (comparison on the parabolic boundary).*

Then $\Phi(x,t) \le v(x,t)$ for all $x \in \mathbb{R}$, $t \in (0,T)$.

Proof. The proof relies on the study of the difference $\Phi - v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$. At the initial time $t = 0$ we have by hypothesis that $\Phi(x,0) - v(x,0) < 0$ for all $x \in \mathbb{R}$.

Now, we argue by contradiction. We assume that the function $\Phi - v$ has a first contact point (x_c, t_c) where $x_c \in \Omega$ and $t_c \in (0,T)$. That is, $(\Phi - v)(x_c,t_c) = 0$ and $(\Phi - v)(x,t) < 0$ for all $0 < t < t_c$, $x \in \mathbb{R}$, by regularity assumptions. Therefore, $(\Phi - v)$ has a global maximum point at (x_c,t_c) on $\mathbb{R} \times (0,t_c]$. Therefore, $v - \Phi$ attains a global minimum at (x_c,t_c).

Since v is a viscosity solution and Φ is an admissible test function then by definition

$$\Phi_t(x_c,t_c) + |\Phi_x(x_c,t_c)|^{m-1}(-\Delta)^\alpha \Phi(x_c,t_c) \ge 0,$$

which is a contradiction since this value is negative by hypothesis. □

4.6.3 Self-Similar Solutions. Formal approach

Self-similar solutions are the key tool in describing the asymptotic behaviour of the solution to certain parabolic problems. We perform here a formal computation of a type of self-similar solution to equation (4.40), being motivated by the construction of suitable lower barriers.

Let $m \in (1,2)$ and $\alpha \in (0,1)$. We search for self-similar solutions to equation (4.40) of the form

$$U(x,t) = \Phi(|y|t^{-b})$$

which solve equation (4.40) in $\mathbb{R} \times (0,\infty)$. After a formal computation, it follows that the exponent $b > 0$ is given by $b = 1/(m-1+2\alpha)$ and the profile function Φ is a solution of the equation

$$by\Phi'(y) - |\Phi'(y)|^{m-1}(-\Delta)^{\alpha}\Phi(y) = 0.$$

We deduce that any possible behaviour of the form $\Phi(y) = c|y|^{-\gamma}$ with $\gamma > 1$ is given by

$$\gamma = \frac{2\alpha + m}{2-m}. \tag{4.45}$$

The value of the self-similarity exponent will be used in the next section for the construction of a lower barrier. A further analysis of self-similar solutions is beyond the purpose of this paper and can be the subject of a new work. We mention that in the case $m = 2$, the profile function Φ has been computed explicitly by Biler, Karch and Monneau in [22].

4.6.4 Positivity estimates

According to the theory on nonlinear parabolic equations, it is to be expected the following result on uniform positivity.

Proposition 4.6.3. *Let v be the solution of Problem* (4.40)-(4.42) *with initial data v_0 satisfying* (4.43)*. Then $v(x,t) \geq k > 0$ in a compact set $K \subset \mathbb{R}$ and for all $t \in [t_1,t_2]$ with $0 < t_1 < t_2$.*

The proof of this result is still a work in progress. We refer to [95].

4.6.5 Construction of the lower barrier

In this section we present a class of sub-solutions of equation (4.40) which represent an important tool in the proof of the infinite speed of propagation. For a suitable choice of parameters this type of sub-solution will give us a lower bound for u in the corresponding domain. This motivates us to refer to this function as a lower barrier. We mention that a similar lower barrier has been constructed in [97].

Let $\gamma = \dfrac{m+2\alpha}{2-m}$ and $b = \dfrac{1}{m-1+2\alpha}$ be the exponents deduced in Section 4.6.3.

We fix $x_0 < 0$. In the sequel we will use as an important tool a function $G : \mathbb{R} \to \mathbb{R}$ such that, given any two constants $C_1 > 0$ and $C_2 > 0$, we have that

- (G1) G is compactly supported in the interval $(-x_0, \infty)$;
- (G2) $G(x) \le C_1$ for all $x \in \mathbb{R}$;
- (G3) $(-\Delta)^s G(x) \le -C_2 |x|^{-(1+2s)}$ for all $x < x_0$.

This technical result will be proven in Lemma 4.7.1 of Section 4.7 (Appendix).

We can now state the main result for the model (4.40) which in particular implies the infinite speed of propagation of model (4.1) for $1 < m < 2$ in dimension $N = 1$.

Theorem 4.6.2 (Infinite speed of propagation). *Let v be the solution of Problem* (4.40)-(4.42) *with initial data v_0 satisfying* (4.43). *Then $v(x,t) > 0$ for all $t > 0$ and $x < x_0$.*

The proof relies on the construction of a lower barrier of the form

$$\Phi_\epsilon(x,t) = (t+1)^{b\gamma} \left((|x| + \xi)^{-\gamma} + G(x) \right) - \epsilon, \quad t \ge 0,\ x \in \mathbb{R},$$

where G satisfies assumptions (G1)-(G2)-(G3) with parameters C_2 given by (4.48) and C_1 to be chosen later. We can also choose $\epsilon > 0$ and $\xi > 0$ in order to obtain the desired lower barrier.

Lemma 4.6.1 (Lower Barrier). *Let $x_0 < 0$, $\epsilon > 0$ and $\xi > 0$. Also, let G be a function with the properties (G1),(G2) and (G3). We consider the barrier*

$$\Phi_\epsilon(x,t) = (t+\tau)^{b\gamma} \left((|x| + \xi)^{-\gamma} + G(x) \right) - \epsilon, \quad t \ge 0,\ x \in \mathbb{R}. \tag{4.46}$$

Then for a suitable choice of the parameter $C_2 > 0$, the function Φ_ϵ satisfies

$$(\Phi_\epsilon)_t + |(\Phi_\epsilon)_x|^{m-1} (-\Delta)^\alpha \Phi_\epsilon \le 0 \quad \textit{for } x < x_0,\ t > 0. \tag{4.47}$$

Moreover, C_1 is a free parameter and $C_2 = C_2(N, m, \alpha, \tau)$.

Proof. We start by checking under which conditions Φ_ϵ satisfies (4.47), that is, Φ is a classical sub-solution of equation (4.40) in Q. To this aim, we have that

$$\begin{aligned}(\Phi_\epsilon)_t + |(\Phi_\epsilon)_x|^{m-1} (-\Delta)^\alpha \Phi_\epsilon &= b\gamma \frac{(t+\tau)^{b\gamma-1}}{(|x|+\xi)^\gamma} + \gamma^{m-1} \frac{(t+\tau)^{b\gamma(m-1)}}{(|x|+\xi)^{(\gamma+1)(m-1)}} (-\Delta)^\alpha \Phi_\epsilon(x,t) \\ &= b\gamma \frac{(t+\tau)^{b\gamma-1}}{(|x|+\xi)^\gamma} + \gamma^{m-1} \frac{(t+\tau)^{b\gamma(m-1)}}{(|x|+\xi)^{(\gamma+1)(m-1)}} (t+\tau)^{b\gamma} \Big((-\Delta)^\alpha [(|x|+\xi)^{-\gamma}] + (-\Delta)^\alpha G \Big).\end{aligned}$$

Now, by Lemma 4.7.2 we get the estimate $(-\Delta)^\alpha \left((|x|+\xi)^{-\gamma} \right) \le C_3 |x|^{-(1+2\alpha)}$ for all $|x| \ge |x_0|$, with positive constant $C_3 = C_3(N, m, \alpha)$. At this step, we choose the parameter C_2 in the assumption (G2) to be at least $C_2 > C_3$. The precise choice will be deduced later. Since $\gamma = (\gamma+1)(m-1)+1+2\alpha$, we continue as follows:

$$\begin{aligned}
&(\Phi_\epsilon)_t + |(\Phi_\epsilon)_x|^{m-1}(-\Delta)^\alpha \Phi_\epsilon \\
&\quad \le b\gamma \frac{(t+\tau)^{b\gamma-1}}{(|x|+\xi)^\gamma} + \gamma^{m-1}\frac{(t+\tau)^{b\gamma m}}{(|x|+\xi)^{(\gamma+1)(m-1)}}(C_3 - C_2)|x|^{-(1+2\alpha)} \\
&\quad = (|x|+\xi)^{-(\gamma+1)(m-1)} \cdot \\
&\qquad \cdot \Big(b\gamma (t+\tau)^{b\gamma-1}(|x|+\xi)^{-(1+2\alpha)} + \gamma^{m-1}(t+\tau)^{b\gamma m}(C_3-C_2)|x|^{-(1+2\alpha)} \Big) \\
&\quad \le (|x|+\xi)^{-(\gamma+1)(m-1)}|x|^{-(1+2\alpha)} \Big(b\gamma (t+\tau)^{b\gamma-1} + \gamma^{m-1}(t+\tau)^{b\gamma m}(C_3 - C_2) \Big)
\end{aligned}$$

which is negative for all $(x,t) \in Q$, if we ensure that C_2 is such that:

$$C_2 > C_3 + b\gamma^{2-m}\tau^{b\gamma(1-m)-1}. \tag{4.48}$$

This choice of C_2 is independent on the parameters $\xi,\ \epsilon$.

□

From now on, we will take $\tau = 1$, which will be enough for our purpose.

Proof. (of Theorem 4.6.2) Let $x_0 < 0$ fixed. We prove that $v(x,t) > 0$ for all $t > 0$ and $x < x_0$. By scaling arguments, it is sufficient to prove the result for typical initial data of the form

$$v_0(x) = H_{x_0}(x) = \begin{cases} 0, & x < x_0, \\ 1, & x > x_0. \end{cases} \tag{4.49}$$

We will prove that $v(x,t) \ge \Phi_\epsilon(x,t)$ in the parabolic domain $Q_T = \{x < x_0,\ t \in [0,T]\}$ by using as an essential tool the Parabolic Comparison Principle established in Proposition 4.6.2. We describe the proof in the graphics below, where the barrier function is represented, for simplicity, without the modification caused by the function $G(\cdot)$ (Figure 4.3).

To this aim we check the required conditions in order to apply the above mentioned comparison result.

• **Comparison on the parabolic boundary.** This will be done in two steps.

(a) **Comparison at the initial time.** The initial data (4.49) naturally impose the following conditions on Φ_ϵ. At time $t = 0$ we have $\Phi_\epsilon(x_0, 0) < 0 = v_0(x_0)$, which holds only if ξ satisfies

$$\xi > x_0 + \epsilon^{-\frac{1}{\gamma}}. \tag{4.50}$$

(b) **Comparison on the lateral boundary.** The positivity estimates given by Proposition 4.6.3 allow us to take $k_1 := \min\{v(x,t) :\ x \ge x_0,\ 0 < t \le T\}$ with $k_1 > 0$. We impose the condition

$$\Phi_\epsilon(x,t) < v(x,t) \quad \text{for all } x \ge x_0,\ t \in [0,T].$$

It is sufficient to have

$$(T+1)^{b\gamma}(\xi^{-\gamma}+C_1) < k_1.$$

The maximum value of T for which this inequality holds is

$$T < \left(\frac{k_1}{\xi^{-\gamma}+C_1}\right)^{1/b\gamma} - 1. \tag{4.51}$$

We need to impose a compatibility condition on the parameters in order to have $T > 0$, that is:

$$\xi > (k_1 - C_1)^{-\frac{1}{\gamma}}. \tag{4.52}$$

The remaining parameter C_1 from assumption (G2) is chosen here such that: $C_1 < k_1$.

By Proposition 4.6.2 we obtain the desired comparison

$$v(x,t) \geq \Phi_\epsilon(x,t) \quad \text{for all } (x,t) \in Q_T.$$

• **Infinite speed of propagation**. Let $x_1 < x_0$ and $t_1 \in (0,T)$ where T is given by (4.51). We prove there exists a suitable choice of ξ and ϵ such that $\Phi_\epsilon(x_1,t_1) > 0$. This is equivalent to impose the following upper bound on ξ:

$$\xi < x_1 + (t_1+1)^b \epsilon^{-\frac{1}{\gamma}}. \tag{4.53}$$

We need to check now if there exists $\epsilon > 0$ such that condition (4.53) is compatible with conditions (4.50) and (4.52). For the compatibility of conditions (4.50) and (4.52) we have

$$x_0 + \epsilon^{-\frac{1}{\gamma}} < \xi < x_1 + (t_1+1)^b \epsilon^{-\frac{1}{\gamma}},$$

that is,

$$\epsilon < \left[\frac{(t_1+1)^b - 1}{x_0 - x_1}\right]^\gamma. \tag{4.54}$$

For conditions (4.52) and (4.53) we need

$$(k_1 - C_1)^{-\frac{1}{\gamma}} \leq \xi < x_1 + (t_1+1)^b \epsilon^{-\frac{1}{\gamma}},$$

which is equivalent to

$$\epsilon < \left[\frac{(t_1+1)^b}{(k_1 - C_1)^{-\frac{1}{\gamma}} - x_1}\right]^\gamma. \tag{4.55}$$

Both upper bounds (4.54) and (4.55) make sense since $0 > x_0 > x_1$ and $k_1 > C_1$.

Summary. The proof was performed in a constructive manner and we summarize it as follows: $C_1 < k_1$, T given by (4.51). Then by taking ϵ the minimum of (4.54)-(4.55), ξ satisfying (4.50)-(4.52)-(4.53) we obtain that $\Phi(t_1,x_1) > 0$.

This proofs that $v(t_1,x_1) > 0$ for any $t \in (0,T)$.

□

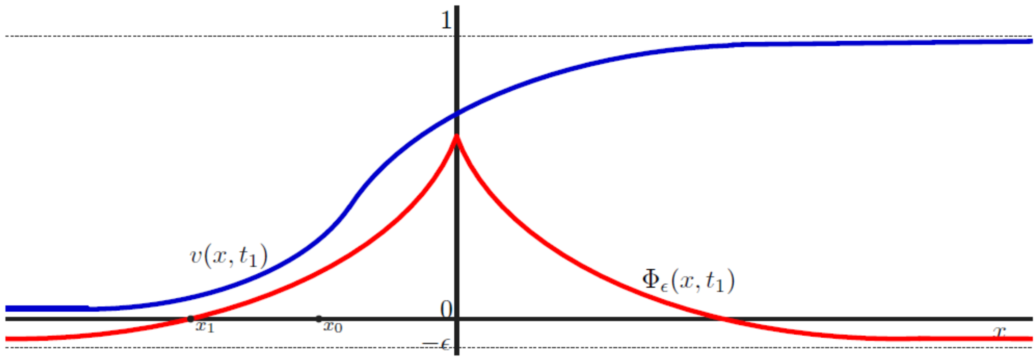


FIGURE 4.3: Comparison with the barrier at time $t > 0$

Remark. The parameter ξ of the barrier depends on ϵ by (4.50) and (4.53) and therefore $\xi \to \infty$ as $\epsilon \to 0$. Therefore $\Phi_\epsilon(x,t) \to 0$ as $\epsilon \to 0$ for every $(x,t) \in Q_T$ and we can not derive a lower parabolic estimate for $v(x,t)$ in Q_T.

4.7 Appendix D

4.7.1 Estimating the Fractional Laplacian

In this section we are interested in estimating the fractional Laplacian of given functions. The definition of $(-\Delta)^s$ was given in Section 2.5.3.

First, given the expression of the fractional Laplacian, we construct a function with the desired properties.

Lemma 4.7.1. *Given two arbitrary constants $C_1, C_2 > 0$ there exists a function $G : \mathbb{R} \to [0,+\infty)$ with the following properties:*

1. *G is compactly supported.*
2. *$G(x) \le C_1$ for all $x \in \mathbb{R}$*
3. *$(-\Delta)^s G(x) \le -C_2|x|^{-(1+2s)}$ for all $x \in \mathbb{R}$ with $d(x, supp(G)) \ge 1$.*

Proof. Let R be an arbitrary positive number to be chosen later. We consider a smooth function $G_1 : \mathbb{R} \to [0,+\infty)$ such that $G_1(x) \le C_1$ for all $x \in \mathbb{R}$ and supported in the interval $[-1,1]$.

We define $G_R(x) = G_1(x/R)$. Therefore $\|G_R\|_{L^1(\mathbb{R})} = R\|G_1\|_{L^1(\mathbb{R})}$, $G_R \le C_1$ and G is supported in the interval $[-R, R]$. Then for $|x| \ge R+1$ we have that

$$\begin{aligned}
(-\Delta)^s G_R(x) &= \sigma_s \int_{\mathbb{R}} \frac{G_R(x) - G_R(y)}{|x-y|^{1+2s}} dy = -\sigma_s \int_{-R}^{R} \frac{G_R(y)}{|x-y|^{1+2s}} dy \\
&\le -\sigma_s \int_{-R}^{R} \frac{G_R(y)}{(|x|+R)^{1+2s}} dy = -\sigma_s (|x|+R)^{-(1+2s)} \|G_R\|_{L^1(\mathbb{R})} \\
&\le -\sigma_s 2^{-(1+2s)} \|G_R\|_{L^1(\mathbb{R})} |x|^{-(1+2s)} = -\sigma_s 2^{-(1+2s)} R \|G_1\|_{L^1(\mathbb{R})} |x|^{-(1+2s)}.
\end{aligned}$$

It is enough to choose $R \ge \dfrac{C_2 2^{1+2s}}{\sigma_s ||G_1||_{L^1(\mathbb{R})}}$ to get $(-\Delta)^s G_R(x) \le C_2 |x|^{-(1+2s)}$. Note that R implicitly depends on C_1 since $||G_1||_{L^1(\mathbb{R})} \le 2C_1$.

□

Secondly, we need to estimate the fractional Laplacian of a negative power function. The following result is similar to one proven by Bonforte and Vázquez in Lemma 2.1 from [28] with the main difference that our function is C^2 away from the origin. We make a brief adaptation of their proof to our situation.

Lemma 4.7.2. *Let $\varphi : \mathbb{R} \to (0,\infty)$, $\varphi = (|x| + \xi)^{-\gamma}$, where $\gamma > 1$ and $\xi > 0$. Then, for all $|x| \ge |x_0| > 1$, we have that*

$$|(-\Delta)^s \varphi(x)| \le \frac{C}{|x|^{1+2s}}, \tag{4.56}$$

with positive constant $C > 0$ that depends only on γ, ξ, s.

Proof. Following the ideas of [28] Lemma 2.1, the computation of the $(-\Delta)^s \varphi(x)$ is based on estimating the integrals on the regions

$$R_1 = \{y : \ |y| > 3|x|/2\}, \quad R_2 = \left\{ y : \ \frac{|x|}{2} < |y| < \frac{3|x|}{2} \right\} \setminus B_{\frac{|x|}{2}}(x),$$

$$R_3 = \{y : \ |x-y| < |x|/2\}, \quad R_4 = \{y : \ |y| < |x|/2\}.$$

Therefore

$$(-\Delta)^s \varphi(x) = \int_{R_1 \cup R_2 \cup R_3 \cup R_4} \frac{\varphi(x) - \varphi(y)}{|x-y|^{1+2s}} dy.$$

We proceed with the estimate of each of the four integrals:

$$I = \int_{|y|>3|x|/2} \frac{\varphi(x)-\varphi(y)}{|x-y|^{1+2s}}dy \le \omega_d \varphi(x) \int_{3|x|/2}^{\infty} \frac{dr}{r^{1+2s}} = \frac{K_1}{|x|^{\gamma+2s}}, \quad \text{where } K_1 = K_1(\gamma, s),$$

$$II = \int_{R_2} \frac{\varphi(x)-\varphi(y)}{|x-y|^{1+2s}}dy \le \frac{\varphi(x)}{(|x|/2)^{1+2s}} \int_{|x|/2}^{3|x|/2} dr = \frac{K_2}{|x|^{\gamma+2s}}, \quad \text{where } K_2 = K_2(\gamma, s),$$

$$III = \int_{R_3} \frac{\varphi(x)-\varphi(y)}{|x-y|^{1+2s}}dy \le \|\varphi''\|_{L^\infty(B_{|x|/2}(x))} \int_{|x-y|\le|x|/2} \frac{1}{|x-y|^{2s-1}}dy$$

$$\le \frac{K_3'}{|x|^{\gamma+2}} \int_0^{|x|/2} \frac{1}{r^{2s-1}}dr \le \frac{K_3}{|x|^{\gamma+2s}}, \qquad \text{where } K_3 = K_3(\gamma, s),$$

$$IV \le \int_{|y|<|x|/2} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{1+2s}}dy \le \left(\frac{2}{|x|}\right)^{1+2s} \int_{|y|<|x|/2} \varphi(y)dy = \frac{K_4}{|x|^{1+2s}},$$

where $K_4 = K_4(\gamma, s, \xi)$.

$\square$

4.7.2 Compact sets in the space $L^p(0,T;B)$

Necessary and sufficient conditions for the convergence in the spaces $L^p(0,T;B)$ are given by Simon in [93]. We recall now their applications to evolution problems. We consider the spaces $X \subset B \subset Y$ with compact embedding $X \to B$.

Lemma 4.7.3. *Let $\mathcal{F}$ be a bounded family of functions in $L^p(0,T;X)$, where $1 \le p < \infty$ and $\partial\mathcal{F}/\partial t = \{\partial f/\partial t : f \in \mathcal{F}\}$ be bounded in $L^1(0,T;Y)$. Then the family $\mathcal{F}$ is relatively compact in $L^p(0,T;B)$.*

Lemma 4.7.4. *Let $\mathcal{F}$ be a bounded family of functions in $L^\infty(0,T;X)$ and $\partial\mathcal{F}/\partial t$ be bounded in $L^r(0,T;Y)$, where $r > 1$. Then the family $\mathcal{F}$ is relatively compact in $C(0,T;B)$.*

4.8 Comments and open problems

• In this dissertation we gave the proof of the existence result as a limit of solutions to the approximate problems in case $m \in (2,3)$. The proof of the existence result for $m \ge 3$ is still under study now and will appear in a forthcoming paper [95] in collaboration with J.L. Vázquez and F. del Teso. The results concerning the finite propagation work also for $m \ge 3$, as stated throughout the proofs.

• **Explicit solutions.** Y. Huang reports [68] the explicit expression of the Barenblatt solution for the special value of m, $m_{ex} = (N+6s-2)/(N+2s)$. The profile is given by

$$F_M(y) = \lambda\,(R^2 + |y|^2)^{-(N+2s)/2},$$

where the two constants λ and R are determined by the total mass M of the solution and the parameter β. Note that for $s = 1/2$ we have $m_{ex} = 1$, and the solution corresponds to the linear case, $u_t = (-\Delta)^{1/2}u$, $F_{1/2}(r) = C(a^2 + r^2)^{-(N+1)/2}$.

Different generalizations of model (CV) are worth studying:

• Changing-sign solutions for the problem $\partial_t u = \nabla \cdot (|u|\nabla P)$, $\quad P = (-\Delta)^{-s}u$.

• Starting from the Problem (CV), an alternative is to consider the problem

$$u_t = \nabla \cdot (|u|\nabla(-\Delta)^{-s}(|u|^{m-2}u)), \quad x \in \mathbb{R}^N, \ t > 0,$$

with $m > 1$. This problem has been studied by Biler, Imbert and Karch in [20, 21]. They construct explicit compactly supported self-similar solutions which generalize the Barenblatt profiles of the PME. They do not prove the finite propagation of a general solution.

• We should consider combining the above models into $\partial_t u = \nabla(|u|^{m-1}\nabla P)$,
$P = (-\Delta)^{-s}u$.
When $s = 0$ and $m = 2$ we obtain the signed porous medium equation $\partial_t u = \Delta(|u|^{m-1}u)$.

Chapter 5

Transformations of self-similar solutions for FPME and PMFP

The present chapter is part of the paper [96] in collaboration with Juan Luis Vázquez and Félix del Teso. This paper [96] is a more complex work involving four different models of nonlocal partial differential equations for which we prove transformation formulas between self-similar solutions. In what follows, I will present the results related to the equations discussed in the previous chapters.

We consider models (FPME) and (PMFP) presented in Chapters 2 and 4:

$$u_t + (-\Delta)^s u^m = 0, \tag{FPME}$$

$$v_t = \nabla \cdot (v^{\widetilde{m}-1} \nabla (-\Delta)^{-\widetilde{s}} v). \tag{PMFP}$$

The behaviour of the solutions of these basic models may be very different depending both on the equation and on the parameters $m, \widetilde{m}$. An efficient way of studying such differences is via the existence and properties of special solutions having particular symmetries, since such solutions are either explicit or semi-explicit, or at least can be analyzed in great detail; the interest is also due to the fact that they are important in describing the properties of much wider classes of solutions. This applies in particular to the class of self-similar solutions, namely, solutions of the type

$$\begin{aligned} u(x,t) &= t^{-\alpha} \phi(x t^{-\beta}), && \text{(type I)} \\ u(x,t) &= (T-t)^{\alpha} \phi(x\,(T-t)^{-\beta}), && \text{(type II)} \\ v(x,t) &= e^{-\gamma t} F(y), \quad y = x e^{-ct}. && \text{(type III)} \end{aligned}$$

The importance of self-similar solutions in the areas of PDEs and Applied Mathematics is attested in a wide literature, cf. Barenblatt's monograph [13] or [105].

5.1 Self-similar solutions

5.1.1 Self-similar solutions for the FPME

The (FPME) has been presented in Chapter 2. For convenience, we recall some useful facts concerning the self-similar solutions, already introduced in Section 2.1.

The long time behaviour of a general solution of the (FPME) is described by the self-similar solutions of type I with finite mass constructed in [107], where it is shown that the equation admits a family of self-similar solutions, also called Barenblatt type solutions, of the form

$$u(x,t) = t^{-N\beta_1}\phi_1(y), \quad y = x\, t^{-\beta_1}\,,$$

and $\beta_1 = 1/(N(m-1)+2s)$. Existence is proved when $m > m_c$, so that β_1 is well-defined and positive. The extra condition that is used to obtain these solutions is $\int u(x,t)\,dx =$ constant in time. This formula produces a solution to equation (FPME) if the profile function ϕ_1 satisfies the following equation

$$(-\Delta)^s \phi_1^m = \beta_1 \nabla\cdot(y\,\phi_1). \tag{5.1}$$

It is proved in the above reference that the profile $\phi_1(y)$ is a smooth and positive function in $\mathbb{R}^N$, it is a radial function, it is monotone decreasing in $r = |y|$ and has a definite decay rate as $|y| \to \infty$, that depends on m as described in Theorem 2.1.2.

A main practical question that remains partially open is to determine if the profile ϕ_1 can be expressed as an explicit or semi-explicit function of $r = |x|$ (and the parameters s and N). The answer is yes in the special case $m = 1$ where the solution is explicit for $s = 1/2$, semi-explicit otherwise. Recently, Huang [68] has shown that for every $s \in (0,1)$ there exists a certain $m_{ex}(s) > m_1$ for which the profile has an explicit expression. More precisely, $m_{ex}(s) = (N+2-2s)/(N+2s)$. For $s = 1/2$ we have $m_{ex}(s) = 1$, thus recovering the formula of the linear fractional heat equation.

For $m < (N-2s)/N$, model (FPME) admits self-similar solutions of type II, as proved by Vázquez and Volzone in [110]. Here we consider only the (FPME) with $m > (N-2s)/N$ and (PMFP) with a corresponding $\widetilde{m}$ interval.

5.1.2 Preliminaries on Model PMFP

We recall some useful facts on self-similar solutions for the *Porous Medium Equation with Fractional Pressure* (PMFP).

• **Case** $m = 2$**.** The study of the problem has been done by Caffarelli and Vázquez [39, 40] and also with Soria [38] in the more natural case $\widetilde{m} = 2$. Previous analysis in 1D is due to Biler et al. [22]. It is proved that for non-negative initial data $u_0 \geq 0$, $u_0 \in L^1(\mathbb{R}^N)$, there exists a non-negative solution $u(x,t) \geq 0$. However, uniqueness of the constructed weak solutions has not been proved but for the case $N = 1$. Moreover, the assumption of having compact support on the initial data implies that the same property for all positive times, $u(\cdot,t)$ is compactly supported for all $t>0$. The existence of a self-similar solution that will be responsible for the asymptotic behaviour is obtained in [20, 21] in 1D and in [40] in all dimensions as the solution of a fractional obstacle problem. The explicit formula for this solution was given in [20], and takes the form

$$v(x,t)=t^{-N//N+2-2s}\Phi(xt^{-1/(N+2-2s)}), \qquad \Phi(y)=(a-b|y|^2)_+^{1-s} \tag{5.2}$$

for suitable constants $a,b>0$.

• **General** $m>1$**.** In Chapter 4 we have presented the theory for general $m>1$ according to [21, 39, 40, 94, 95]. For non-negative initial data $u_0\ge 0$, there exists a non-negative solution $u(x,t)\ge 0$. Different results on the positivity properties have been obtained depending on the parameter m as follows:

- When $N\ge 1$, $s\in(0,1)$, $u_0\ge 0$ compactly supported and $\widetilde{m}\in[2,\infty)$ then the solution $u(x,t)$ is compactly supported for all $t>0$, that is the model has finite speed of propagation.

- When $N=1$, $s\in(0,1)$, $\widetilde{m}\in(1,2)$ and $u_0\ge 0$ then the solution satisfies $u(x,t)>0$ a.e. in $\mathbb{R}$, therefore the model has infinite speed of propagation.

5.1.3 Self-similarity for Model PMFP

We find two main types of self-similar solutions for model (PMFP) depending on the range of the parameter $\widetilde{m}$. The first type are functions that are positive for all times, while second type are functions that extinguish in finite time, separated by a transition type.

• **Self-similarity of first type. Solutions that exist for all positive times.** Arguing in the same way as for the FPME in Section 5.1.1, or the case $\widetilde{m}=2$ of (PMFP) described above, a self-similar function of the first type $v(x,t)$ is a solution to equation (PMFP) conserving mass if

$$v(x,t)=t^{-\alpha_2}\phi_2(y), \qquad y=x\,t^{-\beta_2} \tag{5.3}$$

with

$$\alpha_2=N\beta_2, \quad \beta_2=1/(N(\widetilde{m}-1)+2-2\widetilde{s}),$$

and if the profile function ϕ_2 satisfies the equation

$$\nabla\cdot(\phi_2^{\widetilde{m}-1}\,\nabla(-\Delta)^{-\widetilde{s}}\phi_2)=-\beta_2\nabla\cdot(y\,\phi_2). \tag{5.4}$$

The existence and properties of this family of solutions have not been previously studied in the literature, but for the work of Huang ([68]) who has shown the existence of a certain $m(s)$ for each $s\in(0,1)$ for which an explicit solution can be found.

Remark. In the analysis below we find these solutions in the range of parameters where $\beta_2>0$, that is, for $\widetilde{m}>(N-2+2\widetilde{s})/N$.

• **Self-Similarity of second type. Extinction in finite time.** These solutions have the form

$$v(x,t) = (T-t)^{\overline{\alpha}_2}\psi_2\,(y)\,, \quad y = x(T-t)^{\overline{\beta}_2}. \tag{5.5}$$

We make again the choice $\overline{\alpha}_2 = N\overline{\beta}_2$, even if there can not be a justification in terms of mass conservation since the solutions will now extinguish in finite time (the solution to this seeming incompatibility is that the mass will be actually infinite). We use however the rule for a formal consideration: the divergence structure of the resulting profile equation will make the correspondence with Model (FPME) possible.

Let us determine the value of $\overline{\beta}_2$ such that $v(x,t)$ solves the equation of (PMFP). Since

$$v_t(x,t) = -\overline{\beta}_2(T-t)^{N\overline{\beta}_2-1}\nabla\cdot(y\psi_2),$$

$$\nabla\cdot(v^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}v) = (T-t)^{\overline{\beta}_2(N\widetilde{m}-2\widetilde{s}+2)}\nabla\cdot(\psi_2^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}\psi_2),$$

we get the compatibility condition

$$\overline{\beta}_2 = 1/(N(1-\widetilde{m})+2\widetilde{s}-2)\,.$$

The profile ψ_2 has to satisfy the equation

$$\nabla\cdot(\psi_2^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}\psi_2) = \nabla\cdot(y\psi_2). \tag{5.6}$$

Remark. $\overline{\beta}_2 = -\beta_2$, where β_2 is the self-similarity exponent of first type. We argue now in the range of parameters where $\overline{\beta}_2 > 0$, that is $\widetilde{m} < (N-2+2\widetilde{s})/N$.

• **Self-Similarity of third type. Eternal solutions.** There is a borderline case $\widetilde{m} = (N-2+2\widetilde{s})/N$, which is not included in the previous self-similar solutions. Actually, as $m \to (N-2+2\widetilde{s})/N$ we have $1/\beta_2 = 1/\overline{\beta}_2 \to 0$, and therefore self-similar solutions of the first and second type do not apply here. The possibility of self-similar representation comes from the classical porous medium equation (see [107]), where a third type of self-similar solutions of the form

$$v(x,t) = e^{-\gamma t}F(y), \quad y = xe^{-ct}, \tag{5.7}$$

where $c > 0$ is a free parameter (exponential self-similarity, which usually plays a transition role). We choose $\gamma = ct$ in order to have conservation of mass. It is easy to check that

$$v_t(x,t) = -ce^{Nct}\nabla\cdot(yF),$$

$$\nabla\cdot(v^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}v) = e^{-ct(-N\widetilde{m}+2\widetilde{s}-2)}\nabla\cdot(F^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}F).$$

Then, for $m = (N-2+2\widetilde{s})/N$ we get the following profile equation

$$\nabla\cdot(v^{\widetilde{m}-1}\nabla(-\Delta)^{-\widetilde{s}}v) = -c\nabla\cdot(yF). \tag{5.8}$$

Remark. Solutions of this type live backwards and forward in time, they use to be called eternal solutions.

5.2 Transformation formula

We formulate now the relationship that allows to transform the families of mass-conserving self-similar solutions of models (FPME) and (PMFP) into each other, if suitable parameter ranges are prescribed. Actually, the following theorem states that there exists a precise correspondence between the profiles ϕ_1 and ϕ_2, and the parameters $\widetilde{m}$ and m, as well as $\widetilde{s}$ and s.

Theorem 5.2.1. *Let* $m > (N-2s)/N$*,* $s \in (0,1)$ *and let* $\phi_1 \geq 0$ *be a solution to the profile equation* (5.1)*. The following holds:*

(i) *If* $m > N/(N+2s)$ *then*

$$\phi_2(x) = (\beta_1/\beta_2)^{\frac{m}{1-m}} (\phi_1(x))^m \tag{5.9}$$

is a solution to the profile equation (5.4) *if we put* $\widetilde{m} = (2m-1)/m$ *and* $\widetilde{s} = 1-s$*.*

(ii) *If* $m \in ((N-2s)/N, N/(N+2s))$ *then*

$$\psi_2(x) = \left(\beta_1/\overline{\beta}_2\right)^{\frac{m}{1-m}} (\phi_1(x))^m \tag{5.10}$$

is a solution to the profile equation (5.6) *if we put* $\widetilde{m} = (2m-1)/m$ *and* $\widetilde{s} = 1-s$*.*

(iii) *If* $m = N/(N+2s)$ *then*

$$F(x) = (\beta_1/c)^{\frac{N}{2s}} (\phi_1(x))^{\frac{N}{N+2s}} \tag{5.11}$$

is a solution to the profile equation (5.8) *if we put* $\widetilde{m} = (N-2+2\widetilde{s})/N$ *and* $\widetilde{s} = 1-s$*.*

Comments. The case (i) corresponds to exponents β_1 and $\beta_2 > 0$ and produces new self-similar solutions of (PMFP) of type I, i. e., global in time. We see that $\beta_1 > 0$ if $m > (N-2s)/N$, while $\beta_2 > 0$ if $\widetilde{m} > (N-2+2\widetilde{s})/N$. With the relation $\widetilde{m} = (2m-1)/m$, we have $\widetilde{m} > (N-2+2\widetilde{s})/N$ which is equivalent to $m > N/(N+2s)$. This is another important value in the (FPME), identified in [107], and we have $N/(N+2s) > (N-2s)_+/N$. Therefore, by analyzing the parameters m and $\widetilde{m}$ for which $\beta_1 > 0$ and $\beta_2 > 0$ we have to work in the range of parameters $m > N/(N+2s)$.

Option (ii) produces solutions of (PMFP) that extinguish in finite time, starting with solutions of (FPME) that exist globally in time. This is a remarkable phenomenon of change of behaviour.

Proof. (1) Let us write equation (5.1) in terms of ϕ_2, that is, $\phi_1 = (\beta_2/\beta_1)^{\frac{1}{(1-m)}} \phi_2^{\frac{1}{m}}$, and then

$$(-\Delta)^s \phi_2 = \beta_2 \nabla \cdot (y\, \phi_2^{\frac{1}{m}}).$$

Now, we pass to the parameters $\widetilde{m}$ and $\widetilde{s}$ defined by

$$m = \frac{1}{2-\widetilde{m}} \quad \text{and} \quad s = 1-\widetilde{s} \tag{5.12}$$

and we obtain

$$-\Delta(-\Delta)^{-\widetilde{s}}\phi_2 = \beta_2\, \nabla \cdot (y\, \phi_2^{2-\widetilde{m}}).$$

We can express now Δ as $\nabla\cdot\nabla$, integrate once and use the decay at infinity to transform the previous equation into the vector identity

$$\nabla(-\Delta)^{-\widetilde{s}}\phi_2 = -\beta_2\, y\, \phi_2^{2-\widetilde{m}}.$$

We pass now the term $\phi_2^{\widetilde{m}-1}$ to the left hand side, and finally, assuming regularity on ϕ_2 and taking divergence in both sides of the equation, we obtain

$$\nabla \cdot (\phi_2^{\widetilde{m}-1}\, \nabla(-\Delta)^{-\widetilde{s}}\phi_2) = -\beta_2\, \nabla \cdot (y\, \phi_2).$$

The regularity of ϕ_2 follows from the already proved regularity of ϕ_1 ([107]) and the correspondence (5.9). This is an a posteriori argument. In any case, without using the regularity, ϕ_2 is already a weak solution of problem (PMFP).

(2) The proof is similar to the first case. $\square$

Remarks. (i) Relation between the parameters

$$m \in [1,\infty) \longleftrightarrow \widetilde{m} \in [1,2)$$

$$m \in \left(\frac{N}{N+2s}, 1\right) \longleftrightarrow \widetilde{m} \in \left(\frac{N-2s}{N}, 1\right)$$

$$m \in \left(\frac{N-2s}{N}, \frac{N}{N+2s}\right) \longleftrightarrow \widetilde{m} \in \left(\frac{N-4s}{N-2s}, \frac{N-2s}{N}\right)$$

Notice that $m=1$ implies $\widetilde{m}=1$, which is the *Fractional Linear Heat Equation.* Since $\widetilde{m}_c < 1$, some singular cases of equation (PMFP) are covered where $\widetilde{m} < 1$. Thus, for $s = 1/2$ and $N = 2$ we get the whole range $\widetilde{m} \in (0,2)$.

(ii) Conversely we can pass from a triple $(\widetilde{s}, \widetilde{m}, \phi_2)$ for equation (PMFP) to the corresponding triple (s, m, ϕ_1) for equation (FPME) through the relation

$$m = 1/(2-\widetilde{m}), \quad s = 1-\widetilde{s}, \quad \phi_1 = (\beta_2/\beta_1)^{\frac{1}{(1-m)}}\, \phi_2^{\frac{1}{m}}.$$

The following corollary describes the existence ranges and asymptotic behaviour of the self-similar solutions of (PMFP). It comes as a consequence of our Theorem 5.2.1 and the previously known Theorem 2.1.2.

Corolary 5.2.1. *(i) For every $\widetilde{s} \in (0,1)$ and $\widetilde{m} \in ((N-2+2\widetilde{s})/N, 2)$ there is a fundamental solution of equation* (PMFP) *given by the formula* (5.3). *The behaviour at*

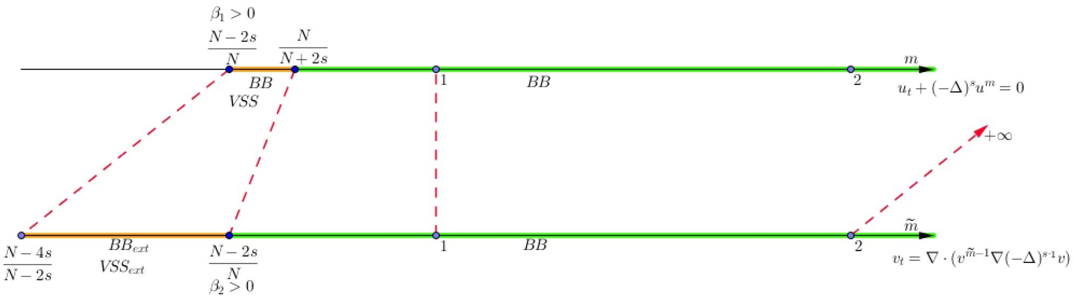


FIGURE 5.1: Related profiles of (FPME) and (PMFP). The picture is drawn for N=2 and $s = \frac{1}{2}$. The notations stand for: BB=Barenblatt solution (type I), BB_{ext}=Barenblatt solution with extinction in finite time (type II), VSS=Very Singular Solution, VVS_{ext}=Very Singular Solution with extinction in finite time.

infinity is given by

$$\phi_2(x) \sim C|x|^{-(N+2-2\widetilde{s})/(2-\widetilde{m})}. \tag{5.13}$$

(ii) For every $\widetilde{s} \in (0,1)$ and $m \in ((N-4+4\widetilde{s})/(N-2+2\widetilde{s}), (N-2+2\widetilde{s})/N)$ there is a finite-time selfsimilar solution of type II given by the formula (5.5) *with the asymptotic behaviour*

$$\psi_2(x) \sim C|x|^{-2(1-\widetilde{s})/(1-\widetilde{m})}. \tag{5.14}$$

(iii) For every $\widetilde{s} \in (0,1)$ and $\widetilde{m} = (N-2+2\widetilde{s})/N$ there is a selfsimilar eternal in time solution of (PMFP) *given by the formula* (5.7)*. The behaviour at infinity is given by*

$$F(x) \sim |x|^{-N}. \tag{5.15}$$

In all cases the self-similar solutions have positive profiles. This is a partial confirmation that equation (PMFP) has infinite speed of propagation for all $\widetilde{m} \in (\widetilde{m}_*, 2)$, $\widetilde{m}_* = (N-4s)/(N-2s) < 1$. In the limit of this interval of infinite propagation we get the case $\widetilde{m} = 2$, i.e., the equation studied in [39] where finite propagation was established. Concerning general classes of solutions, we have proved infinite propagation in [94] for model (PMFP) for $\widetilde{m} \in (1,2)$ in dimension 1. Our corollary amounts to a partial result of infinite propagation in all dimensions for a range of $\widetilde{m}$ that goes below 1.

5.3 Appendix E: Inverse fractional Laplacian and Potentials

The definition of $(-\Delta)^w$ is also done by means of Fourier transform

$$\mathcal{F}((-\Delta)^s f)(\xi) = (2\pi|\xi|)^{2s}\mathcal{F}(f)(\xi),$$

and can be use even for negative values of s. In the range $N/2 < s < 0$ we have an equivalent definition in terms of a Riesz potential

$$(-\Delta)^{-s} f(x) = I_s(f) = \gamma(s)^{-1} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}} dy,$$

acting on functions of the class $\mathcal{S}$. The function γ is defined by

$$\gamma(\rho) = \pi^{N/2}\, 2^{\rho} \frac{\Gamma(\rho/2)}{\Gamma((N-\rho)/2)}.$$

It is well known that the Fourier Transform of the function $f_\alpha(x) = |x|^{-\alpha}$ is

$$\mathcal{F}(f_\alpha)(\xi) = \gamma(N-\alpha)(2\pi)^{\alpha-N}|\xi|^{\alpha-N}.$$

In this way, we can compute $(-\Delta)^{-s} f_\alpha(x)$ as follows,

$$\begin{aligned}
\mathcal{F}((-\Delta)^{-s} f_\alpha)(\xi) &= (2\pi|\xi|)^{-2s}\mathcal{F}(f_\alpha)(\xi) = \gamma(N-\alpha)(2\pi)^{\alpha-N-2s}|\xi|^{\alpha-N-2s} \\
&= \frac{\gamma(N-\alpha)}{\gamma(N-\alpha+2s)}\gamma(N-\alpha+2s)(2\pi)^{\alpha-N-2s}|\xi|^{\alpha-N-2s} \\
&= \frac{\gamma(N-\alpha)}{\gamma(N-\alpha+2s)}\mathcal{F}(f_{\alpha-2s})(\xi),
\end{aligned}$$

that is

$$(-\Delta)^{-s} f_\alpha(x) = \overline{k}(\alpha) f_{\alpha-2s}(x), \qquad \overline{k}(\alpha) = \frac{\gamma(N-\alpha)}{\gamma(N-\alpha+2s)}.$$

More exactly

$$\overline{k}(\alpha) = 2^{-2s}\frac{\Gamma((N-\alpha)/2)\,\Gamma\left((\alpha-2s)/2\right)}{\Gamma(\alpha/2)\,\Gamma((N-\alpha+2s)/2)}.$$

5.4 Comments and open problems

The following questions appear naturally in view of the results of this chapter.

• To decide if the asymptotic behaviour of a general solution of (PMFP) is given by a Barenblatt type solution. This fact is well known for (FPME) for general $m > (N-2s)_+/N$ (see [107]) and for (PMFP) with $\widetilde{m} = 2$ (see [40]).

• To find explicit or semi-explicit formulas for the Barenblatt profiles of models (FPME) and (PMFP).

• To find explicit or semi-explicit solutions of any kind for model (PMFP) with $\widetilde{m} > 2$.

• Is it possible to find a transformation between general solutions of (FPME) and (PMFP)?

• Develop a general theory for the general model $u_t = \nabla(u^m \nabla(-\Delta)^{-s} u^n)$.

Abbreviations

CV	Caffarelli-Vázquez Model $u_t = \nabla \cdot (u \nabla (-\Delta)^{-s} u))$
DNLE	Doubly Nonlinear Equation $u_t = \Delta_p u^m$
DNLE-d	The Dirichlet problem for the Doubly Nonlinear Equation
FHE	Fractional Heat Equation $u_t + (-\Delta)^s u = 0$
FPME	Fractional Porous Medium Equation $u_t + (-\Delta)^s u^m = 0$
HE	Heat Equation $u_t = \Delta u$
KPP	The equation $u_t + (-\Delta)^s u^m = f(u)$ with the reaction term $f(u)$ according to the theory of Kolmogorov, Petrovskii y Piskunov (KPP)
PLE	p-Laplacian Equation $u_t = \Delta_p u$
PME	Porous Medium Equation $u_t = \Delta u^m$
PMFP	Porous Medium with Fractional Pressure $u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u))$
TW	Traveling Wave (Onda Viajera)

Bibliography

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964.

[2] M. Agueh, A. Blanchet, and J.A. Carrillo. Large time asymptotics of the doubly nonlinear equation in the non-displacement convexity regime. *J. Evol. Equ.*, 10(1):59–84, 2010.

[3] Martial Agueh. Asymptotic behavior for doubly degenerate parabolic equations. *C. R. Math. Acad. Sci. Paris*, 337(5):331–336, 2003.

[4] Martial Agueh. Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory. *Adv. Differential Equations*, 10(3):309–360, 2005.

[5] Martial Agueh. Rates of decay to equilibria for p-Laplacian type equations. *Nonlinear Anal.*, 68(7):1909–1927, 2008.

[6] Aomar Anane. Simplicité et isolation de la première valeur propre du p-laplacien avec poids. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(16):725–728, 1987.

[7] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

[8] D. G. Aronson and L. A. Peletier. Large time behaviour of solutions of the Porous Medium Equation in bounded domains. *Journal of Differential Equations*, 39(3):378 – 412, 1981.

[9] D. G. Aronson and H. F. Weinberger. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In *Partial differential equations*

and related topics (Program, Tulane Univ., New Orleans, La., 1974), pages 5–49. Lecture Notes in Math., Vol. 446. Springer, Berlin, 1975.

[10] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.*, 30(1):33–76, 1978.

[11] I. Athanasopoulos and L. A. Caffarelli. Continuity of the temperature in boundary heat control problems. *Adv. Math.*, 224(1):293–315, 2010.

[12] Viorel Barbu. Nonlinear diffusion equations in image processing. *Rev. Roumaine Math. Pures Appl.*, 58(2):175–204, 2013.

[13] G. I. Barenblatt. *Scaling, Self-Similarity, and Intermediate Asymptotics.* Cambridge Univ. Press, Cambridge, 1996. Updated version of Similarity, Self-Similarity, and Intermediate Asymptotics,Consultants Bureau, New York, 1979.

[14] G. I. Barenblatt and J. L. Vázquez. Nonlinear diffusion and image contour enhancement. *Interfaces Free Bound.*, 6(1):31–54, 2004.

[15] G. Barles and C. Imbert. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire.*

[16] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.L. Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.

[17] H. Berestycki, J.M. Roquejoffre, and L. Rossi. The periodic patch model for population dynamics with fractional diffusion. *Discrete Contin. Dyn. Syst. Ser. S*, 4:1–13, 2011.

[18] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1996.

[19] M.F. Bidaut-Véron. Self-similar solutions of the p-laplace heat equation: the case when $p > 2$. *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, 139(01):1–43, 2009.

[20] Piotr Biler, Cyril Imbert, and Grzegorz Karch. Nonlocal porous medium equation: Barenblatt profiles and other weak solutions. *preprint, (2013)* arXiv:1302.7219.

[21] Piotr Biler, Cyril Imbert, and Grzegorz Karch. Barenblatt profiles for a nonlocal porous medium equation. *C. R. Math. Acad. Sci. Paris*, 349(11-12):641–645, 2011.

[22] Piotr Biler, Grzegorz Karch, and Régis Monneau. Nonlinear diffusion of dislocation density and self-similar solutions. *Comm. Math. Phys.*, 294(1):145–168, 2010.

[23] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. *Arch. Ration. Mech. Anal.*, 191(2):347–385, 2009.

[24] R. M. Blumenthal and R. K. Getoor. Some theorems on stable processes. *Trans. Amer. Math. Soc.*, 95:263–273, 1960.

[25] M. Bonforte, J. Dolbeault, G. Grillo, and J. L. Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities. *Proc. Natl. Acad. Sci. USA*, 107(38):16459–16464, 2010.

[26] M. Bonforte, G. Grillo, and J.L. Vázquez. Behaviour near extinction for the Fast Diffusion Equation on bounded domains. *J. Math. Pures Appl. (9)*, 97(1):1–38, 2012.

[27] M. Bonforte, R.G. Iagar, and J.L. Vázquez. Local smoothing effects, positivity, and Harnack inequalities for the fast p-Laplacian equation. *Adv. Math.*, 224(5):2151–2215, 2010.

[28] M. Bonforte and J.L. Vázquez. Quantitative local and global a priori estimates for fractional nonlinear diffusion equations. *Adv. Math.*, 250:242–284, 2014.

[29] Matteo Bonforte and Agnese di Castro. Quantitative local estimates for nonlinear elliptic equations involving $p-$laplacian type operators. *Riv. Mat. Univ. Parma*, 2014. to appear.

[30] Matteo Bonforte and Gabriele Grillo. Super and ultracontractive bounds for doubly nonlinear evolution equations. *Rev. Mat. Iberoam.*, 22(1):111–129, 2006.

[31] Matteo Bonforte, Razvan Gabriel Iagar, and Juan Luis Vázquez. Local smoothing effects, positivity, and Harnack inequalities for the fast p-Laplacian equation. *Adv. Math.*, 224(5):2151–2215, 2010.

[32] Matteo Bonforte and Juan Luis Vázquez. Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations. *Adv. Math.*, 223(2):529–578, 2010.

[33] J. Boussinesq. Recherches théoriques sur l'ecoulement des nappes d'eau infillrees dans le sol et sur le debit de sources. *Comptes Rendus Acad. Sci/J. Math. Pures Appl.*, 10:5–78, 1903.

[34] H. Brezis. Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. *Contributions to Nonlinear Funct. Analysis*, pages 101–150, 1971.

[35] X. Cabré and J.M. Roquejoffre. Propagation de fronts dans les équations de Fisher-KPP avec diffusion fractionnaire. *C. R. Math. Acad. Sci. Paris*, 347(23-24):1361–1366, 2009.

[36] Xavier Cabré, Anne Charline Coulon, and Jean Michel Roquejoffre. Propagation in Fisher–KPP type equations with fractional diffusion in periodic media. *C. R. Math. Acad. Sci. Paris*, 350(19-20):885–890, 2012.

[37] Xavier Cabré and Jean Michel Roquejoffre. Front propagation in Fisher-KPP equations with fractional diffusion. *Comm. Math. Phys.*, 320(3):679–722, 2013.

[38] Luis Caffarelli, Fernando Soria, and Juan Luis Vázquez. Regularity of solutions of the fractional porous medium flow. *J. Eur. Math. Soc. (JEMS)*, 15(5):1701–1746, 2013.

[39] Luis Caffarelli and Juan Luis Vazquez. Nonlinear porous medium flow with fractional potential pressure. *Arch. Ration. Mech. Anal.*, 202(2):537–565, 2011.

[40] Luis A. Caffarelli and Juan Luis Vázquez. Asymptotic behaviour of a porous medium equation with fractional diffusion. *Discrete Contin. Dyn. Syst.*, 29(4):1393–1404, 2011.

[41] J. A. Carrillo and G. Toscani. Asymptotic L^1-decay of solutions of the Porous Medium equation to self-similarity. *Indiana University Mathematics Journal*, 49:113–141, 2000.

[42] J. A. Carrillo and J. L. Vázquez. Fine asymptotics for fast diffusion equations. *Comm. Partial Differential Equations*, 28:1023–1056, 2003.

[43] J.A. Carrillo, Y. Huang, and J.L. Vázquez. in preparation.

[44] Francine Catté, Pierre-Louis Lions, Jean-Michel Morel, and Tomeu Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM J. Numer. Anal.*, 29(1):182–193, 1992.

[45] M. Crandall, H. Ishii, and P.L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992).

[46] A. de Pablo, F. Quirós, A. Rodríguez, and J.L. Vázquez. A fractional porous medium equation. *Adv. Math.*, 226(2):1378–1409, 2011.

[47] A. de Pablo, F. Quirós, A. Rodríguez, and J.L. Vázquez. A general fractional porous medium equation. *Comm. Pure Appl. Math.*, 65(9):1242–1284, 2012.

[48] A. de Pablo, F. Quirós, A. Rodríguez, and J.L. Vázquez. Classical solutions for nonlinear fractional diffusion equations. *preprint*, 2013.

[49] A. de Pablo and A. Sánchez. Travelling wave behaviour for a porous-Fisher equation. *European J. Appl. Math.*, 9(3):285–304, 1998.

[50] A. de Pablo and J.L. Vázquez. Travelling waves and finite propagation in a reaction-diffusion equation. *J. Differential Equations*, 93(1):19–61, 1991.

[51] Manuel Del Pino and Jean Dolbeault. Nonlinear diffusions and optimal constants in Sobolev type inequalities: asymptotic behaviour of equations involving the p-Laplacian. *C. R. Math. Acad. Sci. Paris*, 334(5):365–370, 2002.

[52] Félix del Teso. Finite difference method for a fractional porous medium equation. *Calcolo*, 2013.

[53] Félix del Teso and Juan Luis Vázquez. Finite difference method for a general fractional porous medium equation. `arXiv:1307.2474.`, 2013.

[54] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[55] E.DiBenedetto. *Degenerate Parabolic Equations.* Universitext. Springer-Verlag, New York, 1993.

[56] Albert Einstein. On the movement of small particles suspended in stationary liquids required by the molecular-kinetic theory of heat. *Annalen der Physik*, 17:549–560.

[57] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi. *Higher transcendental functions. Vol. II.* Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1953 original.

[58] E. Feireisl and F. Simondon. Convergence for semilinear degenerate parabolic equations in several space dimensions. *J. Dynam. Differential Equations*, 12(3):647–673, 2000.

[59] P. Felmer and M. Yangari. Fast propagation for fractional kpp equations with slowly decaying initial conditions. *SIAM J. Math. Anal.*, 45:662–678, 2013.

[60] R.A. Fisher. The wave of advance of advantagenous genes. *Ann. Eugenics*, 7:355–369, 1937.

[61] Simona Fornaro, Maria Sosio, and Vincenzo Vespri. $L^r_{loc} - L^\infty_{loc}$ estimates and expansion of positivity for a class of doubly non linear singular parabolic equations. *Discrete Contin. Dyn. Syst. Ser. S*, 7(4):737–760, 2014.

[62] Joseph Fourier. *Théorie analytique de la chaleur.* Éditions Jacques Gabay, Paris, 1988. Reprint of the 1822 original.

[63] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[64] B.H. Gilding and L.A. Peletier. On a class of similarity solutions of the porous media equation. *J. Math. Anal. Appl.*, 55:351–364, 1976.

[65] B.H. Gilding and L.A. Peletier. On a class of similarity solutions of the porous media equation II. *J. Math. Anal. Appl.*, 57:522–538, 1977.

[66] F. Hamel and L. Roques. Fast propagation for kpp equations with slowly decaying initial conditions. *J. Differential Equations*, 249:1726–1745, 2010.

[67] A.K. Head. Dislocation group dynamics iii. Similarity solutions of the continuum approximation. *Philosophical Magazine*, 26.

[68] Y. Huang. Explicit barenblatt profiles for fractional porous medium equations. *arXiv:1312.0469*.

[69] Razvan Iagar. Some results on regularity, self-similarity and asymptotic behavior for nonlinear diffusion equations. *Ph.D. Thesis*, 2010.

[70] R.G. Iagar, A. Sánchez, and J.L. Vázquez. Radial equivalence for the two basic nonlinear degenerate diffusion equations. *J. Math. Pures Appl. (9)*, 89(1):1–24, 2008.

[71] Cyril Imbert, Régis Monneau, and Elisabeth Rouy. Homogenization of first order equations with (u/ϵ)-periodic Hamiltonians. II. Application to dislocations dynamics. *Comm. Partial Differential Equations*, 33(1-3):479–516, 2008.

[72] A. V. Ivanov. Regularity for doubly nonlinear parabolic equations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 209(Voprosy Kvant. Teor. Polya i Statist. Fiz. 12):37–59, 261, 1994.

[73] Espen R. Jakobsen and Kenneth H. Karlsen. A “maximum principle for semicontinuous functions” applicable to integro-partial differential equations. *NoDEA Nonlinear Differential Equations Appl.*, 13(2):137–165, 2006.

[74] S. Kamin, L. A. Peletier, and J. L. Vázquez. On the Barenblatt equation of elastoplastic filtration. *Indiana Univ. Math. J.*, 40(4):1333–1362, 1991.

[75] J. King and P. McCabe. On the Fisher-KPP equation with fast nonlinear diffusion. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 459(2038):2529–2546, 2003.

[76] J. R. King. Interacting dopant diffusions in crystalline silicon. *SIAM J. Appl. Math.*, 48(2):405–415, 1988.

[77] A.N. Kolmogorov, I.G. Petrovskii, and N.S. Piskunov. Etude de l’équation de diffusion avec accroissement de la quantité de matière, et son application à un problème biologique,. *Bjul. Moskowskogo Gos. Univ.*, 17:1–26, 1937.

[78] Tuomo Kuusi and Giuseppe Mingione. Riesz Potentials and Nonlinear Parabolic Equations. *Arch. Ration. Mech. Anal.*, 212(3):727–780, 2014.

[79] Tuomo Kuusi, Juhana Siljander, and José Miguel Urbano. Local Hölder continuity for doubly nonlinear parabolic equations. *Indiana Univ. Math. J.*, 61(1):399–430, 2012.

[80] O. A. Ladyzhenskaja, V. A. Solonnikov, and N. N. Ural'seva. *Linear and quasilinear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[81] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow.* Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[82] N. S. Landkof. *Foundations of modern potential theory.* Springer-Verlag, New York,1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[83] L. S. Leibenzon. The motion of a gas in a porous medium. *Complete Works, vol. 2, Acad. of Science of the USSR, Moscow*, 1930.

[84] Peter Lindqvist. Notes on the p-laplace equation. *lecture notes. http://www.math.ntnu.no/ lqvist/p-laplace.pdf.*

[85] Peter Lindqvist. On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.*, 109(1):157–164, 1990.

[86] J.J. Manfredi and V. Vespri. Large time behavior of solutions to a class of doubly nonlinear parabolic equations. *Electron. J. Differential Equations*, pages No. 02, approx. 17 pp. (electronic only), 1994.

[87] M. Muskat. *The Flow of Homogeneous Fluids Through Porous Media.* McGraw-Hill, New York, 1937.

[88] P. Perona and J. Malik. Scale-space and edge detection using anisotopic diffusion. *IEEE Trans. of Pattern Analysis and Machine Intelligence*, 12:629–639, 1990.

[89] M. M. Porzio and V. Vespri. Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations. *J. Differential Equations*, 103(1):146–178, 1993.

[90] G. Savaré and V. Vespri. The asymptotic profile of solutions of a class of doubly nonlinear equations. *Nonlinear Anal.*, 22(12):1553–1565, 1994.

[91] Sylvia Serfaty and Juan Luis Vázquez. A mean field equation as limit of nonlinear diffusions with fractional Laplacian operators. *Calc. Var. Partial Differential Equations*, 49(3-4):1091–1120, 2014.

[92] J. Serrin. The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Philos. Trans. Roy. Soc. London Ser. A*, 264:413–496, 1969.

[93] J. Simon. Compact sets in the space $l^p(0,t;b)$. *Ann. Mat. Pura Appl.*, 146(4):65–96, 1987.

[94] Diana Stan, Félix del Teso, and Juan Luis Vázquez. Finite and infinite speed of propagation for porous medium equations with fractional pressure. *C. R. Math. Acad. Sci. Paris*, 352(2):123–128, 2014.

[95] Diana Stan, Felix del Teso, and Juan Luis Vázquez. Porous medium equation with fractional potential pressure. *in preparation*, 2014.

[96] Diana Stan, Felix del Teso, and Juan Luis Vázquez. Transformations of self-similar solutions for porous medium equations of fractional type. *submitted,* `arXiv:1402.6877`, 2014.

[97] Diana Stan and Juan Luis Vázquez. The Fisher-KPP equation with nonlinear fractional diffusion. *submitted, arXiv:1303.6823.*

[98] Diana Stan and Juan Luis Vázquez. Asymptotic behaviour of the doubly nonlinear diffusion equation $u_t = \Delta_p u^m$ on bounded domains. *Nonlinear Anal.*, 77:1–32, 2013.

[99] Elias M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[100] Enrico Valdinoci. From the long jump random walk to the fractional laplacian. *Bol. Soc. Esp. Mat. Apl.*, 49:33–44., 2009.

[101] N. Th. Varopoulos. Hardy-Littlewood theory for semigroups. *J. Funct. Anal.*, 63(2):240–260, 1985.

[102] J.L. Vázquez. The mesa problem for the fractional porous medium equation. *preprint,* `arXiv:1403.4866`.

[103] J.L. Vázquez. Asymptotic behaviour for the porous medium equation posed in the whole space. *J. Evol. Equ.*, 3(1):67–118, 2003. Dedicated to Philippe Bénilan.

[104] J.L. Vázquez. The Dirichlet problem for the Porous Medium Equation in bounded domains. Asymptotic behavior. *Monatsh. Math.*, 142(1-2):81–111, 2004.

[105] J.L. Vázquez. *Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type.* Oxford University Press, Oxford, 2006.

[106] J.L. Vázquez. *The Porous Medium Equation. Mathematical Theory.* Oxford University Press, 2007.

[107] Juan Luis Vázquez. Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type. *J. Eur. Math. Soc., to appear,* `arXiv:1205.6332v1`.

[108] Juan Luis Vázquez. Nonlinear diffusion with fractional laplacian operators. *Nonlinear partial differential equations: the Abel Symposium 2010 Holden, Helge and Karlsen, Kenneth H. eds., Springer*, pages 271–298, 2012.

[109] Juan Luis Vázquez. Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators. *Discrete Contin. Dyn. Syst. Ser. S*, 7(4):857–885, 2014.

[110] Juan Luis Vázquez and Bruno Volzone. Optimal estimates for fractional fast diffusion equations. *submitted.* `arXiv:1310.3218`, 2013.

[111] Juan Luis Vázquez and Bruno Volzone. Symmetrization for linear and nonlinear fractional parabolic equations of porous medium type. *To appear in J. Math. Pures Appl.* `arXiv:1303.2970`, 2013.

[112] V. Vespri. On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations. *Manuscripta Math.*, 75(1):65–80, 1992.

[113] Joachim Weickert. A review of nonlinear diffusion filtering. *Scale-Space Theory in Computer Vision Lecture Notes in Computer Science*, 1252:1–28, 1997.

[114] M. Yangari. Fractional reaction diffusion problems. *Ph.D. Thesis*, 2014.